

On the formulation of $D=11$ supergravity and the composite nature of its three-form gauge field

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Abstract

The underlying gauge group structure of the $D = 11$ Cremmer-Julia-Scherk supergravity becomes manifest when its three-form field A_3 is expressed through a set of one-form gauge fields, $B_1^{a_1 a_2}$, $B_1^{a_1 \dots a_5}$, $\eta_{1\alpha}$ and E^a , ψ^α . These are associated with the generators of the elements of a family of enlarged supersymmetry algebras $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$ parametrized by a real number s . We study in detail the composite structure of A_3 extending previous results by D’Auria and Fré, stress the equivalence of the above problem to the trivialization of a standard supersymmetry algebra $\mathfrak{E}^{(11|32)}$ cohomology four-cocycle on the enlarged $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$ superalgebras, and discuss its possible dynamical consequences. To this aim we consider the properties of the first order supergravity action with a composite A_3 field and find the set of extra gauge symmetries that guarantee that the field theoretical degrees of freedom of the theory remain the same as with a fundamental A_3 . The extra gauge symmetries are also present in the so-called rheonomic treatment of the first order $D = 11$ supergravity action when A_3 is composite. Our considerations on the composite structure of A_3 provide one more application of the idea that there exists an extended superspace coordinates/fields correspondence. They also suggest that there is a possible embedding of $D = 11$ supergravity into a theory defined on the enlarged superspace $\tilde{\Sigma}^{(528|32+32)}(s)$.

1 Introduction

Already in the original paper [1] where the standard $D = 11$ supergravity theory was introduced, Cremmer, Julia and Scherk (CJS) considered its possible association with a gauge theory and suggested that the gauge group could be related to $OSp(1|32)$. However, the explicit form of such a connection was unclear as *e.g.*, an İnönü–Wigner contraction of $OSp(1|32)$ did not allow for the spin connection among the set of gauge fields. Also the relation of the three-form gauge field $A_3 = \frac{1}{3!} dx^\mu \wedge dx^\nu \wedge dx^\rho A_{\rho\nu\mu}(x)$ with such a Lie superalgebra was unclear¹. The problem was addressed in [5] where it was found, in particular, that the three-form A_3 of CJS supergravity [1] can be expressed through the graviton $E^a(x) = dx^\mu e_\mu^a(x)$, the gravitino $\psi^\alpha(x) = dx^\mu \psi_\mu^\alpha(x)$, the additional bosonic one-forms $B_1^{ab}(x) = dx^\mu B_\mu^{ab}(x)$, $B_1^{a_1 \dots a_5}(x) = dx^\mu B_\mu^{a_1 \dots a_5}(x)$, and an additional fermionic one-form $\eta_{1\alpha}(x) = dx^\mu \eta_{\mu\alpha}(x)$.

Although the presence of additional fermionic fields is undesirable for the standard eleven-dimensional supersymmetry, the presence of the $\eta_{\mu\alpha}$ field is not a problem in the context of [5]. First, it corresponds to central fermionic generators. Secondly, the additional gravitino-like field $\eta_{\mu\alpha}$ appears in the description of $D = 11$ supergravity only through the three-form field A_3 , which is considered as a composite of the ‘old’ (E^a, ψ^α) and ‘new’ ($B_1^{ab}, B_1^{abcde}, \eta_{1\alpha}$) fields,

$$A_3 = A_3(E^a, \psi^\alpha; B_1^{ab}, B_1^{abcde}, \eta_{1\alpha}); \quad (1.1)$$

the new bosonic fields also appear through A_3 only.

The composite structure in Eq. (1.1) suggests [5] a possible underlying gauge symmetry of the $D = 11$ supergravity. The new fields $B_\mu^{ab}(x)$, $B_\mu^{abcde}(x)$, $\eta_{\mu\alpha}(x)$ may be treated as gauge fields associated with new antisymmetric tensor generators $Z_{ab} = Z_{[ab]}$, $Z_{abcde} = Z_{[abcde]}$ and a new fermionic generator Q'^α , which extends the super–Poincaré algebra in which Q_α , P_a and M_{ab} correspond to the gravitino field $\psi^\alpha = dx^\mu \psi_\mu^\alpha(x)$, the graviton $E^a(x) = dx^\mu e_\mu^a(x)$ and the spin connection $\omega^{ab} = dx^\mu \omega_\mu^{ab}(x)$. The possibility of constructing $A_{\mu\nu\rho}$ from the above set of gauge fields fixes the free differential algebra of $B_1^{ab}(x)$, $B_1^{abcde}(x)$, $\eta_{1\alpha}(x)$ and, hence, the algebra of generators P_a , M_{ab} , Q_α , Z_{ab} , Z_{abcde} , Q'^α [5] (see below and Sec. 4 for details). Two possible superalgebras allowing for a composite nature of A_3 were found in [5]; we will call them ‘D’Auria–Fré superalgebras’. Both of them are central extensions of the M-theory superalgebra or M-algebra [6] (see also [7])²,

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= P_{\alpha\beta}, & [P_{\alpha\beta}, P_{\gamma\delta}] &= 0, \\ P_{\alpha\beta} = P_{\beta\alpha} &= P_a \Gamma_{\alpha\beta}^a + Z_{ab} i \Gamma_{\alpha\beta}^{ab} + Z_{a_1 \dots a_5} \Gamma_{\alpha\beta}^{a_1 \dots a_5}, \end{aligned} \quad (1.2)$$

$$[Q_\alpha, P_{\beta\gamma}] = 0, \quad (1.3)$$

by a new fermionic central charge Q'^α . These algebras are defined by Eqs. (1.2) plus

$$\begin{aligned} [P_a, Q_\alpha] &= \delta(\Gamma_a Q')_\alpha, \\ [Z_{ab}, Q_\alpha] &= i\gamma_1(\Gamma_{ab} Q')_\alpha, \\ [Z_{abcde}, Q_\alpha] &= \gamma_2(\Gamma_{abcde} Q')_\alpha, \\ [Q'^\alpha, \text{all}] &= 0, \end{aligned} \quad (1.4)$$

¹See [2] for an interesting treatment of p -form gauge fields as Goldstone fields of Lie superalgebras, [3] for the corresponding sigma model-like action for supergravity and [4] for a reformulation of 11-dimensional supergravity as a theory of the graviton, the gravitino and an independent spin connection in which A_3 is treated as a composite of E^a , ψ^α and ω^{ab} .

²See [8, 9] and refs. therein for further generalizations of the M-theory superalgebra and for their structure.

for two sets of specific values of the constants $\delta, \gamma_1, \gamma_2$. In general, Eqs. (1.2), (1.4) define a one-parametric family of superalgebras, since the allowed values of constants $\delta, \gamma_1, \gamma_2$ are restricted [5] only by the Jacobi identity

$$\delta + 10\gamma_1 - 6!\gamma_2 = 0; \quad (1.5)$$

one parameter, γ_1 if nonzero and δ otherwise, may be absorbed in the normalization of the central fermionic generator Q'^α and (in this sense) is inessential.

The essential parameter s distinguishing the non-isomorphic members of the family $\tilde{\mathfrak{E}}(s) = \tilde{\mathfrak{E}}^{(528|32+32)}(s)$ [10]³ of the fermionic central extensions (hence $32 + 32$ and not just 64) of the M-algebra can be introduced *e.g.* by parametrizing $\delta, \gamma_1, \gamma_2$ as follows:

$$s := \frac{\delta}{2\gamma_1} - 1 \quad \Rightarrow \quad \begin{cases} \delta = 2\gamma_1(s + 1), \\ \gamma_2 = 2\gamma_1\left(\frac{s}{6!} + \frac{1}{5!}\right). \end{cases} \quad (1.6)$$

(this makes sense for $\gamma_1 \neq 0$; to apply this for $\gamma_1 = 0$ one should consider the limit $\gamma_1 \rightarrow 0$, $s \rightarrow \infty$, $\gamma_1 s = \delta/2 \rightarrow \text{finite}$). The properties of the two specific D'Auria–Fré superalgebras ($\tilde{\mathfrak{E}}(3/2)$ and $\tilde{\mathfrak{E}}(-1)$ in the above notation) did not have a clear origin. This question was taken up in [11] and, in particular, whether these two superalgebras could be contractions of $osp(1|64)$ or $su(32|1)$. The answer was negative, and the authors of [11] noted the possibility of looking at non-semisimple supergroups involving $OSp(1|32)$ in such a context.

Recently we have found [10] that *all the $s \neq 0$ members* of the family $\tilde{\mathfrak{E}}(s)$ allow for a composite A_3 expressed in terms of one-form gauge fields. This implies that $D=11$ supergravity possesses a gauge symmetry under the $\tilde{\Sigma}(s \neq 0) \bowtie SO(1, 10) = \tilde{\Sigma}^{(528|32+32)}(s \neq 0) \bowtie SO(1, 10)$ supergroup associated with the $\tilde{\mathfrak{E}}(s \neq 0) \bowtie so(1, 10)$ Lie superalgebra. This *underlying* gauge symmetry is hidden in the original CJS formulation but becomes manifest in the $D=11$ supergravity with a composite A_3 .

Although the presence of a family of superalgebras $\tilde{\mathfrak{E}}(s \neq 0) \bowtie so(1, 10)$, rather than a unique one, may indicate that the found answer on the hidden gauge group structure of the $D = 11$ supergravity is not the final one, the origin of these hidden symmetry supergroups is now clearer [10]. Firstly, all the corresponding supergroups $\tilde{\Sigma}(s \neq 0) \bowtie SO(1, 10)$ are nontrivial deformations of $\tilde{\Sigma}(0) \bowtie SO(1, 10)$, and the latter, as well as $\tilde{\Sigma}(0) \bowtie Sp(32) \supset \tilde{\Sigma}(0) \bowtie SO(1, 10)$ are *expansions*⁴ of the $OSp(1|32)$ supergroup [10].

In this paper we give further details of the derivation of the above results on the composite structure of the A_3 three-(super)form *i.e.*, of the hidden gauge symmetry of $D = 11$ supergravity under (any of) the $\tilde{\Sigma}(s \neq 0) \bowtie SO(1, 10)$ supergroups, and study some of its possible dynamical consequences. To this end we consider the spacetime (component) approach, the standard superspace one and the intermediate rheonomic [14, 5, 15] approach to CJS $D = 11$ supergravity when the A_3 (super)field is composite. To this aim, we consider the original proposal [5] of substituting Eq. (1.1) for A_3 into the first-order formulation of $D = 11$ supergravity action (also proposed in [5], see also [16]⁵). We find that such a dynamical system

³We shall denote by \mathfrak{E} ($\tilde{\mathfrak{E}}$) the supersymmetry (enlarged supersymmetry) algebras associated with the corresponding rigid superspace supergroups, denoted Σ ($\tilde{\Sigma}$). The symbols Σ , $\tilde{\Sigma}$ will also be used to denote the corresponding non-flat superspaces (in which case there is no group structure) without risk of confusion.

⁴The *expansion method* [12, 13] is a new method of generating new Lie algebras starting from a given algebra. It includes [13] the Inönü–Wigner and generalized contractions as a particular case, but in general leads to algebras of larger dimension than the original one.

⁵The action of [5, 16] is first order both for the gravity part and for the A_3 field. An action that is first order with respect to the gravitational part but second order in A_3 was constructed in [11]; a first order formulation in A_3 but with a composite spin connection was given in [17].

presents ‘extra’ gauge symmetries [denoted ‘extra’ to distinguish them from those associated with $\tilde{\Sigma}(s \neq 0) \otimes SO(1, 10)$]. These make the number of degrees of freedom with a composite A_3 the same as those of the standard CJS supergravity [1].

The extra gauge symmetries resulting from the composite structure of A_3 are also present in the ‘rheonomic’ action [14, 15] for $D=11$ supergravity [5]. This is given by the first order action [5, 16] where all the fields are replaced by superfields and the integration surface is an arbitrary bosonic surface \mathcal{M}^{11} in standard superspace $\Sigma^{(11|32)}$. This composite structure of A_3 makes natural to consider \mathcal{M}^{11} in the rheonomic action as a surface in the enlarged superspace $\tilde{\Sigma}(s) = \tilde{\Sigma}^{(528|32+32)}(s)$. This suggests an embedding of $D = 11$ supergravity into a theory in a $D = 11$ enlarged superspace $\tilde{\Sigma}^{(528|32+32)}(s \neq 0)$. This is supported by observing that, as we stress in this paper, the search for a composite structure for the A_3 field along [5, 10] is equivalent to solving the problem of trivializing a Chevalley–Eilenberg [18] (CE) four-cocycle of the standard supersymmetry algebra $\mathfrak{E} = \mathfrak{E}^{(11|32)}$ cohomology. This requires moving from $\mathfrak{E}^{(11|32)}$ to $\tilde{\mathfrak{E}} = \tilde{\mathfrak{E}}^{(528|32+32)}(s \neq 0)$, the supersymmetry algebra of the rigid enlarged superspace $\tilde{\Sigma}^{(528|32+32)}(s \neq 0)$. In this perspective the composite character of the A_3 field, *i.e.*, the fact that it may be written in terms of one-form fields associated with a larger supersymmetry group, can be considered as a further example of the extended superspace coordinates/(super)fields correspondence⁶.

The paper is organized as follows. In Sec. 2 we present a brief review of the standard superfield (Sec. 2.2), spacetime component (Sec. 2.1, 2.4) and rheonomic (Sec. 2.5) approaches to $D = 11$ CJS supergravity. We point out the rôle of free differential algebras (FDAs) in the supergravity description (Sec. 2.3), describe their relation with *Lie* superalgebras and enlarged superspaces and stress, in this perspective, the peculiarity of $D = 11$ supergravity due to the presence of the three-form field⁷ A_3 . As we discuss in Sec. 3, A_3 cannot be associated with a Maurer–Cartan (MC) form of a Lie algebra; rather, dA_3 is associated with a nontrivial CE four-cocycle of the $\mathfrak{E}^{(11|32)}$ cohomology. In Sec. 4 we give the details of the derivation of our recent result [10] on the expression of A_3 in terms of the one-form gauge fields of a one-parametric family of superalgebras, which are nontrivial deformations of an *expansion* of the $osp(1|32)$ superalgebra denoted $osp(1|32)(2, 3, 2)$ (see [13] for the notation). We stress the equivalence of this problem to that of trivializing the $\mathfrak{E}^{(11|32)}$ CE four-cocycle on the extended algebra $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$, and describe how our family of composite A_3 structures includes the two D’Auria and Fré ones as particular cases. Another member of our family gives a particularly simple form of A_3 that does not involve a five-index one-form gauge field. In Sec. 5 we study the consequences of the composite structure of A_3 for the first order supergravity action (Sec. 5.1) and find a set of extra gauge symmetries which reduces the number of degrees of freedom to those of the action with a fundamental or ‘elementary’ A_3 field. These extra gauge

⁶The idea of field-space democracy is explicitly stated in Berezin [20] (‘supermathematics... contains a hint about the existence of a fundamental symmetry between coordinates and fields’) and is implicit in the work of D. V. Volkov [21]. The field-space *democracy* framework was further discussed in [22] in the context of the Ogievetski–Sokatchev formulation of $D = 4$ $N = 1$ superfield supergravity. The case for a (worldvolume) fields/extended superspace coordinates *correspondence* principle for superbranes has been advocated in [9] (see also [23] in the context of κ -symmetry).

⁷Notice that other higher dimensional supergravities also include higher form fields. For instance, $D=10$ type IIB supergravity includes the RR (Ramond–Ramond) four-form C_4 and two two-form gauge fields, the NS–NS (Neveu–Schwarz–Neveu–Schwarz) two-form B_2 and the RR one C_2 . Thus, our discussions on enlarged superspaces and hidden gauge symmetries are relevant there too.

Notice also the appearance of 32 fermionic additional coordinates (associated with the Green algebra) in a recent analysis of covariant superstring quantization [19].

symmetries are also shown to be present in the rheonomic action (Sec. 5.3) with a composite A_3 , which then can be treated as an integral over an eleven-dimensional bosonic surface in the enlarged superspace $\tilde{\Sigma}^{(528|32+32)}(s)$. This suggests an embedding of $D=11$ supergravity in a theory defined on such an enlarged superspace. Our conclusions are presented in Sec. 6.

2 Free differential algebras, superspace constraints and first order action of $D = 11$ supergravity

2.1 Differential forms in $D = 11$ supergravity

Any formulation of CJS supergravity involves the graviton, $e_\mu^a(x)$, the gravitino $\psi_\mu^\alpha(x)$, and the antisymmetric tensor field $A_{\mu_1\mu_2\mu_3}(x)$, as well as the spin connection $\omega_\mu^{ab}(x)$. This last one is considered to be a composite of physical fields (in the second order approach) or becomes composite on the mass shell (in the first order approach, see [5, 11, 15, 16]). All these fields may be associated with a set of differential forms on $D = 11$ spacetime M^{11}

$$\begin{aligned} E^a(x) &= dx^\mu e_\mu^a(x) , & \psi^\alpha(x) &= dx^\mu \psi_\mu^\alpha(x) , \\ A_3(x) &= \frac{1}{3!} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} A_{\mu_3\mu_2\mu_1}(x) , \\ \omega^{ab}(x) &= dx^\mu \omega_\mu^{ab}(x) . \end{aligned} \tag{2.1}$$

Further one may introduce the gauge field

$$A_6(x) = \frac{1}{6!} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_6} A_{\mu_6\dots\mu_1}(x) ; \tag{2.2}$$

its field strength $F_7(x) = dA_6 + A_3 \wedge dA_3$ is dual to the field strength $F_4(x) = dA_3$ of $A_3(x)$, $F_7(x) = *F_4(x)$ (see Eq. (2.24) below).

2.2 On-shell superspace constraints for $D = 11$ supergravity

The above fields may also be associated with a set of superforms on the standard $D = 11$ superspace $\Sigma^{(11|32)}$ with coordinates $Z^M = (x^\mu, \theta^{\hat{\alpha}})$,

$$\begin{aligned} E^a(Z) &= dZ^M E_M^a(Z) , \\ \psi^\alpha(Z) &:= E^\alpha(Z) = dZ^M E_M^\alpha(Z) , \end{aligned} \tag{2.3}$$

$$E^A := (E^a, E^\alpha) , \tag{2.4}$$

$$A_3(Z) = \frac{1}{3!} dZ^{M_1} \wedge dZ^{M_2} \wedge dZ^{M_3} A_{M_3M_2M_1}(Z) \equiv \frac{1}{3!} E^{A_1} \wedge E^{A_2} \wedge E^{A_3} A_{A_3A_2A_1}(Z) , \tag{2.5}$$

$$\omega^{ab}(Z) = dZ^M \omega_M^{ab}(Z) \equiv E^C \omega_C^{ab}(Z) , \tag{2.6}$$

$$A_6(Z) = \frac{1}{6!} dZ^{M_1} \wedge \dots \wedge dZ^{M_6} A_{M_6\dots M_1}(Z) , \tag{2.7}$$

provided these superform potentials obey the superspace supergravity constraints [24, 25, 26]

$$T^a = -iE^\alpha \wedge E^\beta \Gamma_{\alpha\beta}^a, \quad (2.8)$$

$$T^\alpha = -\frac{i}{18}E^a \wedge E^\beta \left(F_{ac_1c_2c_3} \Gamma^{c_1c_2c_3}_{\alpha\beta} + \frac{1}{8}F^{c_1c_2c_3c_4} \Gamma_{ac_1c_2c_3c_4} \right)_\beta^\alpha + \frac{1}{2}E^a \wedge E^b T_{ba}{}^\alpha(Z), \quad (2.9)$$

$$R^{ab} = E^\alpha \wedge E^\beta \left(-\frac{1}{3}F^{abc_1c_2} \Gamma_{c_1c_2}^{\alpha\beta} + \frac{i}{3 \cdot 5!}(*F)^{abc_1\dots c_5} \Gamma_{c_1\dots c_5}^{\alpha\beta} \right) + E^c \wedge E^\alpha \left(-iT^{ab\beta} \Gamma_{c\beta\alpha} + 2iT_c^{[a\beta} \Gamma^b]_{\beta\alpha} \right) + \frac{1}{2}E^d \wedge E^c R_{cd}{}^{ab}(Z), \quad (2.10)$$

$$\mathcal{F}_4 := dA_3 = \frac{1}{2}E^\alpha \wedge E^\beta \wedge \bar{\Gamma}_{\alpha\beta}^{(2)} + \frac{1}{4!}E^{c_4} \wedge \dots \wedge E^{c_1} F_{c_1\dots c_4}(Z), \quad (2.11)$$

$$\mathcal{F}_7 := dA_6 + A_3 \wedge dA_3 = \frac{i}{2}E^\alpha \wedge E^\beta \wedge \bar{\Gamma}_{\alpha\beta}^{(5)} + \frac{1}{7!}E^{c_7} \wedge \dots \wedge E^{c_1} F_{c_1\dots c_7}(Z). \quad (2.12)$$

In the above Eqs. (2.8)–(2.12) T^a , T^α , R^{ab} are the torsion and curvature two-forms,

$$T^a := DE^a(Z) := dE^a - E^b \wedge \omega_b{}^a, \quad (2.13)$$

$$T^\alpha := DE^\alpha(Z) := dE^\alpha - E^\beta \wedge \omega_\beta{}^\alpha, \quad (2.14)$$

$$R^{ab} := d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b, \quad (2.15)$$

ω^{ab} is the spin connection,

$$\omega_\beta{}^\alpha := \frac{1}{4}\omega^{ab}\Gamma_{ab\beta}{}^\alpha, \quad (2.16)$$

$\mathcal{F}_4 = dA_3$ and $\mathcal{F}_7 = dA_6 + A_3 \wedge dA_3$ are the field strength superforms, and we have used the notation

$$\begin{aligned} \bar{\Gamma}_{\alpha\beta}^{(2)} &:= \frac{1}{2}E^b \wedge E^a \Gamma_{ab\alpha\beta}, \\ \bar{\Gamma}_{\alpha\beta}^{(5)} &:= \frac{1}{5!}E^{a_5} \wedge \dots \wedge E^{a_1} \Gamma_{a_1\dots a_5\alpha\beta}. \end{aligned} \quad (2.17)$$

As discussed in [24, 25, 26], the study of the Bianchi identities

$$DT^a \equiv -E^b \wedge R_b{}^a, \quad (2.18)$$

$$DT^\alpha \equiv -E^\beta \wedge R_\beta{}^\alpha := -\frac{1}{4}E^\beta \wedge R^{ab} \Gamma_{ab\beta}{}^\alpha, \quad (2.19)$$

$$DR^{ab} \equiv 0, \quad (2.20)$$

$$d\mathcal{F}_4 \equiv 0, \quad (2.21)$$

$$d\mathcal{F}_7 - \mathcal{F}_4 \wedge \mathcal{F}_4 \equiv 0, \quad (2.22)$$

shows that the set of constraints (2.8)–(2.12) is consistent provided that the Riemann tensor $R_{cd}{}^{ab}$ and the field strengths of the gravitino ($T_{ab}{}^\alpha$) and of the gauge field ($F_{c_1\dots c_4}(Z)$) obey the (superfield generalizations of the) equations of motion.

Actually, the system of the constraints (2.8)–(2.12) is over-complete. This is indicated by the fact that the gauge field strengths $F_{c_1\dots c_4}(Z)$ already enter in the expressions for the torsion (2.9) and the curvature (2.10) of superspace. Indeed, the torsion constraints (2.8), (2.9) and (2.10) already imply the above mentioned dynamical equations and provide the

automatic consistency of the remaining constraints (2.11), (2.12), as may be seen by studying the Bianchi identities (2.18), (2.19), (2.20) (see [27] for an even stronger result). However, when the differential superforms A_3 and A_6 are introduced, the study of the Bianchi identities simplifies essentially, which provides a shortcut that was already used in the first papers [24, 25]. For instance, studying the Bianchi identities (2.21) with the constraints (2.8), (2.11) and $T^\alpha = E^b \wedge E^\beta T_{\beta b}^\alpha + \frac{1}{2} E^a \wedge E^b T_{ba}^\alpha(Z)$ instead of the specific form of (2.9), one finds, in addition to $D_{[c_5} F_{c_1 c_2 c_3 c_4]} = 0$ (indicating that $F_{c_1 c_2 c_3 c_4}$ is the field strength of a $A_{c_1 c_2 c_3}$) and $D_\alpha F_{c_1 c_2 c_3 c_4} = -3! T_{[c_1 c_2}^\beta \Gamma_{c_3 c_4] \beta}^\alpha$, the equation $T_{(\beta|[a} \gamma \Gamma_{bc]} |\alpha) \gamma} = i/3! F_{abcd} \Gamma_{\alpha\beta}^d$. The solution of this equation expresses $T_{\beta b}^\alpha$ in terms of F_{abcd} as given in Eq. (2.9).

It is especially interesting to see how the gauge field equations appear when also the six-superform A_6 with the field strength \mathcal{F}_7 , Eq. (2.12), is introduced [26]. Studying (2.22), one finds, in addition to $D_\alpha F_{a_1 \dots a_7} = -21 i T_{[a_6 a_7}^\beta \Gamma_{a_1 \dots a_5] \beta}^\alpha$, also the pure bosonic Bianchi identities

$$D_{[c_1} F_{c_2 \dots c_8]} - \frac{7!}{4! 4!} F_{[c_1 \dots c_4} F_{c_5 \dots c_8]} = 0 \quad (2.23)$$

and the duality relation for the bosonic fields strength

$$F_{c_1 \dots c_7} = (*F_4)_{c_1 \dots c_7} := \frac{1}{4!} \varepsilon_{c_1 \dots c_7 b_1 \dots b_4} F^{b_1 \dots b_4} . \quad (2.24)$$

Inserting the duality relations (2.24) into the bosonic Bianchi identities for the dual field strength, Eq. (2.23), the (superfield generalization of the bosonic) equations of motion for the three-form gauge field are found,

$$D_{[c_1} (*F_4)_{c_2 \dots c_8]} - \frac{7!}{4! 4!} F_{[c_1 \dots c_4} F_{c_5 \dots c_8]} = 0 . \quad (2.25)$$

2.3 Free differential algebra of $D = 11$ supergravity

A free differential algebra or FDA [28, 5, 15, 29] (termed Cartan integrable system in [5]) is an exterior algebra with constant coefficients generated by a set of forms that is closed under the action of the exterior differential; the MC one-forms of a Lie algebra generate the simplest FDA. The supergravity constraints, Eqs. (2.8)-(2.12), may be considered as solutions of the equations of a FDA given in terms of differential forms on superspace. To encode these supergravity constraints into a FDA one has to (re)define curvatures in such a way that their definitions include all the terms with derivatives of forms and the wedge products of forms with constant coefficients from Eqs. (2.18)-(2.22) and, instead of specifying the expressions for these curvatures in terms of superfields like F_{abcd} , T_{ab}^α , subject them to Bianchi identities that are solved by the above supergravity constraints. Note that the notion of *abstract* FDA is more general than the set of supergravity constraints to which it gives rise. First, a FDA may be considered as an algebra of forms over spacetime M^{11} (in this case the FDA curvatures were called “supersymmetric” curvatures). But one may also think of it as an abstract FDA, where all the differential forms characteristic of $D = 11$ supergravity

$$E^a , \quad \psi^\alpha , \quad \omega^{ab} , \quad (2.26)$$

$$A_3 , \quad A_6 , \quad (2.27)$$

are treated as independent, abstract forms without specifying the manifold on which they might be defined. For one-forms, this is tantamount to saying that these forms are defined on

a (group) manifold with a number of coordinates equal to the number of independent forms, as in the so-called *group-manifold* or *rheonomic approach* [14, 15, 5].

The FDA of the standard CJS supergravity is defined by the curvatures of the forms in Eqs. (2.26), (2.27) [5]

$$\mathbf{R}^a := DE^a + i\psi^\alpha \wedge \psi^\beta \Gamma_{\alpha\beta}^a, \quad (2.28)$$

$$\mathbf{R}^\alpha := T^\alpha = D\psi^\alpha := d\psi^\alpha - \psi^\beta \wedge \omega_\beta^\alpha, \quad (2.29)$$

$$\mathbf{R}^{ab} := R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c^b, \quad (2.30)$$

$$\mathbf{R}_4 := dA_3 - \frac{1}{2}\psi^\alpha \wedge \psi^\beta \wedge \bar{\Gamma}_{\alpha\beta}^{(2)}, \quad (2.31)$$

$$\mathbf{R}_7 := dA_6 + A_3 \wedge dA_3 - \frac{i}{2}\psi^\alpha \wedge \psi^\beta \wedge \bar{\Gamma}_{\alpha\beta}^{(5)}, \quad (2.32)$$

satisfying the Bianchi identities (2.18)–(2.22), now written in terms of $\mathbf{R}^a, \dots, \mathbf{R}_4, \mathbf{R}_7$,

$$\mathcal{D}\mathbf{R}^a := D\mathbf{R}^a + E^b \wedge \mathbf{R}_b^a - 2i\psi^\alpha \wedge \mathbf{R}^\beta \Gamma_{\alpha\beta}^a \equiv 0, \quad (2.33)$$

$$\mathcal{D}\mathbf{R}^\alpha := D\mathbf{R}^\alpha + \frac{1}{4}\psi^\beta \wedge \mathbf{R}^{ab} \Gamma_{ab\beta}^\alpha \equiv 0, \quad (2.34)$$

$$\mathcal{D}\mathbf{R}^{ab} := D\mathbf{R}^{ab} = 0, \quad (2.35)$$

$$\mathcal{D}\mathbf{R}_4 := d\mathbf{R}_4 + \psi^\alpha \wedge \mathbf{R}^\beta \wedge \bar{\Gamma}_{\alpha\beta}^{(2)} + \frac{1}{2}\psi^\alpha \wedge \psi^\beta \wedge E^b \wedge \mathbf{R}^a \Gamma_{ab\alpha\beta} \equiv 0, \quad (2.36)$$

$$\begin{aligned} \mathcal{D}\mathbf{R}_7 := & d\mathbf{R}_7 - \left(\mathbf{R}_4 + \frac{1}{2}\psi \wedge \psi \wedge \bar{\Gamma}^{(2)} \right) \wedge \left(\mathbf{R}_4 + \frac{1}{2}\psi \wedge \psi \wedge \bar{\Gamma}^{(2)} \right) - \\ & - i\psi^\alpha \wedge \mathbf{R}^\beta \wedge \bar{\Gamma}_{\alpha\beta}^{(5)} + \frac{i}{2 \cdot 4!} \psi^\alpha \wedge \psi^\beta \wedge E^{c_4} \wedge \dots \wedge E^{c_1} \wedge \mathbf{R}^a \Gamma_{ac_1 \dots c_4 \alpha\beta} + \\ & + \frac{1}{4} \psi^\alpha \wedge \psi^\beta \wedge \psi^\gamma \wedge \psi^\delta \wedge \bar{\Gamma}_{\alpha\beta}^{(2)} \wedge \bar{\Gamma}_{\gamma\delta}^{(2)} \equiv 0. \end{aligned} \quad (2.37)$$

In this abstract FDA framework, the counterpart of the complete set of the superspace constraints Eqs. (2.8)–(2.12) can be written as

$$\mathbf{R}^a = 0, \quad (2.38)$$

$$\mathbf{R}_4 = F_4 := \frac{1}{4!} E^{c_4} \wedge \dots \wedge E^{c_1} F_{c_1 \dots c_4}, \quad (2.39)$$

$$\mathbf{R}_7 = F_7 := \frac{1}{7!} E^{c_7} \wedge \dots \wedge E^{c_1} F_{c_1 \dots c_7}, \quad (2.40)$$

plus more complicated expressions for $\mathbf{R}^\alpha = T^\alpha$ and $\mathbf{R}^{ab} = R^{ab}$, Eqs. (2.9), (2.10), which can be shortened introducing the notation

$$t_{1\beta}^\alpha = E^b t_{b\beta}^\alpha := \frac{i}{18} E^a \left(F_{ac_1 c_2 c_3} \Gamma^{c_1 c_2 c_3}_a + \frac{1}{8} F^{c_1 c_2 c_3 c_4} \Gamma_{ac_1 c_2 c_3 c_4} \right)_\beta^\alpha, \quad (2.41)$$

in which case they read

$$\mathbf{R}^\alpha := T^\alpha = \psi^\beta \wedge t_{1\beta}^\alpha + \frac{1}{2} E^a \wedge E^b T_{ba}^\alpha, \quad (2.42)$$

$$\begin{aligned} \mathbf{R}^{ab} := & R^{ab} = 2i\psi^\alpha \wedge \psi^\beta t_{(\alpha}^\gamma \Gamma_{\beta\gamma)}^{ab} - E^a \wedge \psi^\alpha \left(iT^{ab\beta} \Gamma_{c\beta\alpha} - 2iT_c^{[a\beta} \Gamma^{b]}_{\beta\alpha} \right) + \\ & + \frac{1}{2} E^d \wedge E^c R_{cd}^{ab}. \end{aligned} \quad (2.43)$$

Eqs. (2.38)–(2.40), (2.42), (2.43) may be looked at as a solution of the Bianchi identities (2.33)–(2.37). Their pull-back to spacetime M^{11} or to a bosonic arbitrary eleven-dimensional surface $\mathcal{M}^{11} \subset \Sigma^{(11|32)}$ in superspace or, even, in a larger supergroup manifold with more coordinates, can be obtained from the group-manifold or rheonomic action, $S = \int_{\mathcal{M}^{11}} \mathcal{L}_{11}[E^a, \psi^\alpha, \omega^{ab}, A_3, F_{a_1 a_2 a_3 a_4}]$ [5], which we discuss now.

2.4 First order action for CJS supergravity with ‘elementary’ A_3 field

2.4.1 First order component action

The first order action for CJS $D = 11$ supergravity,

$$S = \int_{M^{11}} \mathcal{L}_{11}[E^a, \psi^\alpha, \omega^{ab}, A_3, F_{a_1 a_2 a_3 a_4}] \quad , \quad (2.44)$$

is the integral over eleven-dimensional spacetime M^{11} of the eleven-form \mathcal{L}_{11} [5, 16]

$$\begin{aligned} \mathcal{L}_{11} = & \frac{1}{4} R^{ab} \wedge E_{ab}^{\wedge 9} - D\psi^\alpha \wedge \psi^\beta \wedge \bar{\Gamma}_{\alpha\beta}^{(8)} + \frac{1}{4} \psi^\alpha \wedge \psi^\beta \wedge (T^a + i/2 \psi \wedge \psi \Gamma^a) \wedge E_a \wedge \bar{\Gamma}_{\alpha\beta}^{(6)} + \\ & + (dA_3 - a_4) \wedge (*F_4 + b_7) + \frac{1}{2} a_4 \wedge b_7 - \frac{1}{2} F_4 \wedge *F_4 - \frac{1}{3} A_3 \wedge dA_3 \wedge dA_3 \quad , \end{aligned} \quad (2.45)$$

where, following [16], we have denoted

$$a_4 := \frac{1}{2} \psi^\alpha \wedge \psi^\beta \wedge \bar{\Gamma}_{\alpha\beta}^{(2)} := -\frac{1}{4} \psi^\alpha \wedge \psi^\beta \wedge E^a \wedge E^b \Gamma_{ab\alpha\beta} \quad , \quad (2.46)$$

$$b_7 := \frac{i}{2} \psi^\alpha \wedge \psi^\beta \wedge \bar{\Gamma}_{\alpha\beta}^{(5)} := \frac{i}{2 \cdot 5!} \psi^\alpha \wedge \psi^\beta \wedge E^{a_1} \wedge \dots \wedge E^{a_5} \Gamma_{a_1 \dots a_5 \alpha\beta} \quad , \quad (2.47)$$

and introduced *purely bosonic* forms $F_4, *F_4$, constructed from the auxiliary (zero-form) antisymmetric tensor F_{abcd} (see also (2.39), (2.40))

$$F_4 := \frac{1}{4!} E^{a_4} \wedge \dots \wedge E^{a_1} F_{a_1 \dots a_4} \quad , \quad (2.48)$$

$$*F_4 := -\frac{1}{4!} E_{a_1 \dots a_4}^{\wedge 7} F^{a_1 \dots a_4} \equiv \frac{1}{7!4!} E^{b_7} \wedge \dots \wedge E^{b_1} \varepsilon_{b_1 \dots b_7 a_1 \dots a_4} F^{a_1 \dots a_4} \quad . \quad (2.49)$$

We also use the compact notation (see Eq.(2.17))

$$\bar{\Gamma}_{\alpha\beta}^{(k)} := \frac{1}{k!} E^{a_k} \wedge \dots \wedge E^{a_1} \Gamma_{a_1 \dots a_k \alpha\beta} := \frac{(-1)^{k(k-1)/2}}{k!} \bar{\Gamma}_{\alpha}^{(1)} \beta_1 \wedge \bar{\Gamma}_{\beta_1}^{(1)} \beta_2 \wedge \dots \wedge \bar{\Gamma}_{\beta_{k-1}}^{(1)} \beta \quad (2.50)$$

and

$$E_{a_1 \dots a_k}^{\wedge(11-k)} := \frac{1}{(11-k)!} \varepsilon_{a_1 \dots a_k b_1 \dots b_{11-k}} E^{b_1} \wedge \dots \wedge E^{b_{11-k}} \quad . \quad (2.51)$$

(In the notation of [16], $E_{a_1 \dots a_k}^{\wedge(11-k)} := \Sigma_{a_1 \dots a_k}$ and $\bar{\Gamma}_{\alpha\beta}^{(k)} := (-)^{k(k-1)/2} \gamma_{\alpha\beta}^{(k)}$).

We remark that the Hodge star defined on the purely bosonic form/tensors does not produce any problem in extending (2.45) to an eleven-superform on superspace. This allows for a ‘rheonomic’ treatment of the action (2.44), (2.45) [14, 5, 15]. In it, the Lagrangian form (2.45) is defined on a standard $D = 11$ superspace or even on a larger ‘supergroup manifold’ and M^{11} becomes an arbitrary bosonic surface \mathcal{M}^{11} in that manifold. See Sec. 2.5 for further discussion.

2.4.2 Equations of motion for A_3 and F_{abcd}

Let us denote the eight-form appearing as the variation of the action (2.44) with respect to A_3 by \mathcal{G}_8 , *i.e.*

$$\delta_A S = \int \mathcal{G}_8 \wedge \delta A_3, \quad \frac{\delta S}{\delta A_3} := \mathcal{G}_8. \quad (2.52)$$

From Eqs. (2.44), (2.45) one reads

$$\mathcal{G}_8 = d(*F_4 + b_7 - A_3 \wedge dA_3), \quad (2.53)$$

and thus the equation of motion for free supergravity in differential form is

$$\mathcal{G}_8 = d(*F_4 + b_7 - A_3 \wedge dA_3) = 0. \quad (2.54)$$

This includes the auxiliary field $F_{abcd} = F_{[abcd]}$ (see (2.49)) which on the mass shell is identified with the covariant field strengths [5, 16]. Indeed, the variation of the action with respect to this field has the form

$$\delta_F S = \int (dA_3 - a_4 - F_4) \wedge * \delta F_4 = \quad (2.55)$$

$$= -\frac{1}{4!} \int (dA_3 - a_4 - F_4) \wedge E_{a_1 \dots a_4}^{\wedge 7} \delta F^{a_1 \dots a_4}. \quad (2.56)$$

Hence, the equation of motion $\delta S / \delta F^{a_1 \dots a_4} = 0$ can be written as

$$* \frac{\delta S}{\delta F_4} = (dA_3 - a_4 - F_4) = 0. \quad (2.57)$$

Notice that Eq. (2.57) (see Eq. (2.46)) formally coincides with the FDA relations (2.31) after the solution of Bianchi identities (2.39) is used.

2.4.3 Other equations of motion

The variation of the action (2.44), (2.45) with respect to the spin connection gives

$$\frac{\delta S_{11}}{\delta \omega^{ab}} = \frac{1}{4} E_{abc}^{\wedge 8} \wedge (T^c + i\psi^\alpha \wedge \psi^\beta \Gamma_{\alpha\beta}^c) = 0 \quad \Rightarrow \quad T^a := DE^a = -i\psi^\alpha \wedge \psi^\beta \Gamma_{\alpha\beta}^a. \quad (2.58)$$

This clearly gives the pull-back of the FDA relation (2.28) with (2.38) for the forms defined on M^{11} (or defined on a larger superspace but pulled back on M^{11}). Taking in mind the algebraic equations (2.58), (2.57), one finds that the fermionic equation for CJS supergravity may be written in the compact form [16]

$$\frac{\delta S_{11}}{\delta \psi^\alpha} = 0 \quad \Rightarrow \quad \Psi_{10\beta} := \hat{D}\psi^\alpha \wedge \bar{\Gamma}_{\alpha\beta}^{(8)} = 0, \quad (2.59)$$

in terms of a generalized holonomy connection [30, 31] (see Eq. (2.41) above)

$$\hat{D}\psi^\alpha := d\psi^\alpha - \psi^\beta \wedge w_\beta^\alpha \equiv d\psi^\alpha - \psi^\beta \wedge (\omega_\beta^\alpha + t_{1\beta}^\alpha), \quad (2.60)$$

$$w_\beta^\alpha := \omega_\beta^\alpha + t_{1\beta}^\alpha, \quad \omega_\beta^\alpha = \frac{1}{4}\omega^{ab}\Gamma_{ab\beta}^\alpha, \quad (2.61)$$

$$t_{1\beta}^\alpha = \frac{i}{18}E^a \left(F_{ab_1b_2b_3}\Gamma^{b_1b_2b_3}{}_\beta{}^\alpha + \frac{1}{8}F^{b_1b_2b_3b_4}\Gamma_{ab_1b_2b_3b_4\beta}{}^\alpha \right). \quad (2.62)$$

The explicit form of the Einstein equation for $D = 11$ supergravity

$$M_{10a} := R^{bc} \wedge E_{abc}^{\wedge 8} + \dots = 0 \quad (2.63)$$

will not be needed in this paper. It can be found in [16] in similar differential form notation.

2.5 Rheonomic approach and ‘generalized action principle’: a way from first order component action to superspace supergravity

2.5.1 The rheonomic action for supergravity as gauge equivalent to the first order component action

As known already from [5], the action (2.44), (2.45) may give rise to the so-called rheonomic action [14, 15]. This allows, starting from a component first order action, to arrive at the set of superspace supergravity constraints (see also [32] for a brief selfcontained discussion).

The *rheonomic action* is obtained by replacing in the action (2.44), (2.45) all the forms on spacetime $E^a(x)$, $\psi^\alpha(x)$, $A_3(x)$, $\omega^{ab}(x)$ (including zero-forms or fields $F_{abcd}(x)$) by superforms (superfields) on the standard superspace $\Sigma^{(11|32)}$, Eqs. (2.3), (2.4), (2.5), (2.6), taken on a bosonic eleven-dimensional surface \mathcal{M}^{11} in $\Sigma^{(11|32)}$,

$$\mathcal{M}^{11} : Z^M = \tilde{Z}^M(x^\mu) = (x^\mu, \tilde{\theta}^{\tilde{\alpha}}(x)), \quad (2.64)$$

$$\left\{ \begin{array}{c} E^a(x) \\ \psi^\alpha(x) \\ A_3(x) \\ F_{abcd}(x) \end{array} \right\} \mapsto \left\{ \begin{array}{c} E^a(\tilde{Z}(x)) \\ \psi^\alpha(\tilde{Z}(x)) \\ A_3(\tilde{Z}(x)) \\ F_{abcd}(\tilde{Z}(x)) \end{array} \right\} = \left\{ \begin{array}{c} E^a(x, \tilde{\theta}(x)) \\ E^\alpha(x, \tilde{\theta}(x)) \\ A_3(x, \tilde{\theta}(x)) \\ F_{abcd}(x, \tilde{\theta}(x)) \end{array} \right\}. \quad (2.65)$$

In this way, the rheonomic action of the $D = 11$ supergravity (see [5, 15]) is given by

$$S_{11}^{rh} = \int_{\mathcal{M}^{11}} \mathcal{L}_{11}(x^\mu, \theta^{\tilde{\alpha}}) := \int_{\mathcal{M}^{11}} \mathcal{L}_{11}(x^\mu, \tilde{\theta}^{\tilde{\alpha}}(x)), \quad (2.66)$$

where $\mathcal{L}_{11}(Z^M) \equiv \mathcal{L}_{11}(x^\mu, \theta^{\tilde{\alpha}})$ is given by Eq. (2.45), where all the forms on spacetime are replaced by the superforms (2.3), (2.4), (2.5), (2.6) on the standard superspace, $\mathcal{L}_{11}(\tilde{Z}^M(x)) \equiv \mathcal{L}_{11}(x^\mu, \tilde{\theta}^{\tilde{\alpha}}(x))$ is the pull-back of $\mathcal{L}_{11}(Z^M)$ by the map $x \mapsto \tilde{Z}(x)$ in (2.64) to the spacetime \mathcal{M}^{11} . Thus, in essence, a ‘rheonomic’ action is given by the integral of a differential D -form defined on a D -dimensional surface embedded in a larger manifold (here superspace) and where not only the fields, but also the surface itself, are varied.

In principle, one could also consider \mathcal{M}^{11} embedded into the superPoincaré group manifold, thus including the Lorentz group coordinates; the number of independent one-forms and the number of coordinates would then coincide. However, this would give nothing new in

our perspective⁸. For the standard $D = 11$ supergravity a complete correspondence between different differential forms and coordinates is impossible due to the independent three-form field A_3 . Thus, the apparent lack of gauge theory treatment of $D = 11$ supergravity makes unfeasible to complete the ‘rheonomic’ or ‘group manifold’ programme of [14] for this case. This becomes possible if, following [5], one expresses A_3 in terms of products of one-forms. We will discuss this in Sec. 5.

Varying this ‘rheonomic action’ (2.66) with respect to differential forms one obtains a set of equations like Eqs. (2.54), (2.57), (2.58), (2.59), (2.63), but for forms replaced by the superforms on the surface \mathcal{M}^{11} of Eq. (2.64),

$$dA_3(\tilde{Z}(x)) = a_4(\tilde{Z}(x)) + F_4(\tilde{Z}(x)) , \quad (2.67)$$

$$T^a(\tilde{Z}(x)) = -iE^\alpha(\tilde{Z}(x)) \wedge E^\beta(\tilde{Z}(x)) \Gamma_{\alpha\beta}^a , \quad (2.68)$$

$$\mathcal{G}_8(\tilde{Z}(x)) = 0 , \quad (2.69)$$

$$\Psi_{10\alpha}(\tilde{Z}(x)) = 0 , \quad M_{10\ a}(\tilde{Z}(x)) = 0 . \quad (2.70)$$

Eqs. (2.67), (2.68) are just the expressions of the supergravity constraints (2.11) and (2.8) on the surface \mathcal{M}^{11} . In Eq. (2.67), $a_4(\tilde{Z}(x))$ is the four-form a_4 (2.46) on \mathcal{M}^{11} ,

$$a_4(\tilde{Z}(x)) = \frac{1}{2} E^\alpha(\tilde{Z}(x)) \wedge E^\beta(\tilde{Z}(x)) \wedge \bar{\Gamma}_{\alpha\beta}^{(2)}(\tilde{Z}(x)) . \quad (2.71)$$

The action (2.66) also involves the fermionic field $\tilde{\theta}^{\tilde{\alpha}}(x)$ specifying the surface $\mathcal{M}^{11} \subset \Sigma^{(11|32)}$, Eq. (2.64). This field is treated as a dynamical variable, on the same footing as the differential (super)forms E^a etc. In the original articles [14, 15] (see also [5]) this corresponds to the statement that the surface \mathcal{M}^{11} itself is varied as the differential form fields are. Thus,

$$\delta\mathcal{M}^{11} \leftrightarrow \delta\tilde{\theta}^{\tilde{\alpha}}(x) , \quad (2.72)$$

and the complete variation of the rheonomic action (2.66) reads

$$\begin{aligned} \delta S_{11}^{rh} &= \int_{\mathcal{M}^{11}} \delta\mathcal{L}_{11}(Z) + \int_{\delta\mathcal{M}^{11}} \mathcal{L}_{11}(Z) \\ &\equiv \int_{M^{11}} \delta\mathcal{L}_{11}(Z)|_{Z^M=\tilde{Z}^M(x)} + \int_{M^{11}} \frac{\partial\mathcal{L}_{11}(x, \tilde{\theta}(x))}{\partial\tilde{\theta}^{\tilde{\alpha}}(x)} \delta\tilde{\theta}^{\tilde{\alpha}}(x) . \end{aligned} \quad (2.73)$$

The complete set of the equations of motion that follow from the rheonomic action (2.66) includes, in addition to (2.67), (2.68), (2.69), (2.70), the Euler-Lagrange equation for the fermionic field $\tilde{\theta}^{\tilde{\alpha}}(x)$. It is given by

$$\frac{\delta S_{11}^{rh}}{\delta\tilde{\theta}^{\tilde{\alpha}}(x)} := \frac{\partial\mathcal{L}_{11}(x, \tilde{\theta}(x))}{\partial\tilde{\theta}^{\tilde{\alpha}}(x)} = 0 . \quad (2.74)$$

However, this new equation (2.74) is satisfied identically when Eqs. (2.67)–(2.70) are taken into account (see [14, 15]). To see this one first notices that the variation of the Lagrangian

⁸The inclusion of the Lorentz group coordinates might be, however, relevant in a different context as *e.g.*, in the search for a formulation of higher dimensional supergravity in Lorentz-harmonic superspace (see [33, 34]) in a way similar to the original ‘internal’ harmonic superspace approach of [35].

form \mathcal{L}_{11} can be written as a Lie derivative $L_\delta = di_\delta + i_\delta d$, where i_δ is the inner product with respect to the vector field that determines the variation. It satisfies Leibniz's rule,

$$i_\delta(\Omega_q \wedge \Omega_p) = \Omega_q \wedge i_\delta \Omega_p + (-)^p (i_\delta \Omega_q) \wedge \Omega_p, \quad (2.75)$$

for any $p(q)$ -form Ω_p (Ω_q) (as the one-forms E^a , ψ^α , and the three-form A_3). The variation of the rheonomic action (2.73) is given by the pull-back of

$$\delta \mathcal{L}_{11}(Z) = i_\delta d \mathcal{L}_{11}(Z) + d(i_\delta \mathcal{L}_{11}(Z)) \quad (2.76)$$

where the second term may be ignored in δS_{11}^{rh} when the surface \mathcal{M}^{11} has no boundary, $\partial \mathcal{M}^{11} = \emptyset$ [notice that this is not the case for Hořava–Witten heterotic M-theory [36], but we do not consider this case here]. Thus, $\delta S_{11}^{rh} = \int_{\mathcal{M}^{11}} i_\delta d \mathcal{L}_{11}(Z) + \int_{\mathcal{M}^{11}} \partial \mathcal{L}_{11}(x, \tilde{\theta}(x)) / \partial \tilde{\theta}^{\tilde{\alpha}}(x) \delta \tilde{\theta}^{\tilde{\alpha}}(x)$ and all the equations of motion (2.67), (2.68), (2.69), (2.70) follow from

$$i_\delta d \mathcal{L}_{11}(Z)|_{\mathcal{M}^{11}} = 0. \quad (2.77)$$

Reciprocally, Eq. (2.77) is satisfied *for any variation* δ if the equations of motion (2.67), (2.68), (2.69), (2.70) are taken into account. Now, the second term in (2.73) comes from a particular fermionic general coordinate transformation of the superform, $\mathcal{L}_{11}(x, \theta + \delta\theta) - \mathcal{L}_{11}(x, \theta)$ on \mathcal{M}^{11} , and, hence, is also given by the Lie derivative, $\mathcal{L}_{11}(x, \theta + \delta\theta) - \mathcal{L}_{11}(x, \theta) = -(i_\delta d + di_\delta) \mathcal{L}_{11}(x, \theta)$, but now with respect to the vector field defining the variation $\delta \tilde{\theta}^{\tilde{\alpha}}(x)$ of the fermionic field $\tilde{\theta}^{\tilde{\alpha}}(x)$,

$$\int \frac{\partial \mathcal{L}_{11}(x, \tilde{\theta}(x))}{\partial \tilde{\theta}^{\tilde{\alpha}}(x)} \delta \tilde{\theta}^{\tilde{\alpha}}(x) = \int i_{\delta \tilde{\theta}^{\tilde{\alpha}}(x)} d \mathcal{L}_{11}(x, \tilde{\theta}(x)). \quad (2.78)$$

As a result, Eq. (2.74) reads (*cf.* Eq. (2.77))

$$\frac{\delta S_{11}^{rh}}{\delta \tilde{\theta}^{\tilde{\alpha}}(x)} \delta \tilde{\theta}^{\tilde{\alpha}}(x) = i_{\delta \tilde{\theta}^{\tilde{\alpha}}(x)} d \mathcal{L}_{11}(x, \tilde{\theta}(x)) = 0 \quad (2.79)$$

so that it is satisfied identically due to (2.77) *i.e.*, when the equations of motion (2.67)–(2.70) are taken into account. This dependence of the equation of motion (2.79) is just the Noether identity reflecting the existence of a gauge symmetry that acts additively on the fermionic function $\tilde{\theta}^{\tilde{\alpha}}(x)$, $\delta \tilde{\theta}^{\tilde{\alpha}}(x) = \beta^{\tilde{\alpha}}(x)$. This fermionic gauge symmetry is the symmetry under arbitrary ‘deformations’ (changes) of the bosonic surface \mathcal{M}^{11} in superspace,

$$\delta \tilde{\theta}^{\tilde{\alpha}}(x) = \beta^{\tilde{\alpha}}(x) \Leftrightarrow \delta \mathcal{M}^{11} = \text{arbitrary}. \quad (2.80)$$

This independence on the location of the bosonic surface \mathcal{M}^{11} in $\Sigma^{(11|32)}$ is the basis for the formulation of the rheonomic or “*generalized action*” principle.

Let us stress that, although the above proof uses on-shell arguments (usual in the language of the second Noether theorem), as Eq. (2.77) collects the equations of motion (2.67)–(2.70), the variation (2.80) provides a true gauge symmetry of the action (2.66)⁹. In particular the transformations of this symmetry can be used to set $\tilde{\theta}^{\tilde{\alpha}}(x) = 0$ *i.e.*, to identify \mathcal{M}^{11} with the bosonic body M^{11} of superspace. Hence one sees that the rheonomic action of supergravity is gauge equivalent to the component first order action defined in terms of the spacetime component fields.

⁹What happens is that the variation of the $\tilde{\theta}^{\tilde{\alpha}}(x)$, which enters as a parameter of superforms, is compensated by the appropriate variations of the functions (E_M^a *etc.*) in these superforms; see [32] for further discussion.

2.5.2 Generalized action principle: rheonomic action plus ‘lifting’ to superspace

What is new in the rheonomic action with respect to the component one is that it produces equations that are valid on an arbitrary surface \mathcal{M}^{11} in standard superspace. Moreover, it has a gauge symmetry that allows for arbitrary changes of this surface, Eq. (2.80). As a result the (differential (super)form) equations of motion, Eqs. (2.67)–(2.70), are valid on an *arbitrary* surface in superspace. Furthermore, since the set of all these surfaces span the whole superspace, this suggests that one may try to extend or ‘lift’ these equations from a surface \mathcal{M}^{11} in superspace to superspace itself, *i.e.* to substitute the fermionic superspace coordinates θ and $d\theta$ for the fermionic coordinate functions $\tilde{\theta}(x)$ and $d\tilde{\theta}(x)$ in them. This procedure, the so-called rheonomic *lifting*, is not a consequence of the rheonomic action (2.66), but rather an additional step, the consistency of which needs to be checked. Although such a consistency is not guaranteed (see [32] and refs. therein for a discussion), it works for the standard CJS supergravity: lifting Eqs. (2.67), (2.68) to the standard superspace one arrives at the standard superspace $\Sigma^{(11|32)}$ constraints, Eqs. (2.11) and (2.8). All other constraints (2.9), (2.10) as well as the consequences of the superfield relations that follows from the lifting of the dynamical equations (2.69), (2.70) can be reproduced by checking the consistency of the superspace constraints (2.8) and (2.11), *i.e.* by studying the Bianchi identities (see Sec 2.2).

Thus, and precisely in this sense, the rheonomic action (2.66) provides a bridge between the spacetime component action and the standard superspace formulation of supergravity. Considered together with the second step of rheonomic lifting this action leads to the ‘generalized action principle’ [14] (see also [37]) which reproduces the $D = 11$ superspace supergravity constraints. In Sec. 5 we will see that the rheonomic action with a composite A_3 (Sec. 4) leads us naturally to an enlarged superspace with additional bosonic and fermionic coordinates in the sense that \mathcal{M}^{11} may be understood as an arbitrary surface in this superspace. But before turning to the composite structure of A_3 , let us discuss the relation of the FDA in Sec. 2.3 with the rigid standard superspace, the supergroup manifold of the supertranslation group, and show that the A_3 (super)form is the potential three-form of a nontrivial Chevalley–Eilenberg four-cocycle on the standard supersymmetry algebra $\mathfrak{E}^{(11|32)}$.

3 Rigid superspaces as supergroup manifolds: From FDA to Lie algebras, CE cocycles and enlarged superspaces

To make clear the relation of a FDA with a supergroup manifold, let us consider the one defined by (2.28), (2.29), (2.30) setting the curvatures equal to zero, $\mathbf{R}^a = 0$, $\mathbf{R}^\alpha = 0$, $\mathbf{R}^{ab} = 0$. The resulting equations are the Maurer–Cartan (MC) equations of the superPoincaré algebra

$$dE^a = E^b \wedge \omega_b^a - i\psi^\alpha \wedge \psi^\beta \Gamma_{\alpha\beta}^a, \quad (3.1)$$

$$d\psi^\alpha = \psi^\beta \wedge \omega_\beta^\alpha, \quad (3.2)$$

$$d\omega^{ab} = \omega^{ac} \wedge \omega_c^b. \quad (3.3)$$

One may easily solve these equations by

$$\begin{aligned} \omega^{ab} &= 0, & \psi^\alpha &= \Pi^\alpha = d\theta^{\tilde{\alpha}} \delta_{\tilde{\alpha}}^\alpha := d\theta^\alpha, \\ E^a &= \Pi^a := dx^\mu \delta_\mu^a - id\theta \Gamma^a \theta; \end{aligned} \quad (3.4)$$

where Π^a , Π^α are the (MC) one-forms of the standard supersymmetry algebra $\mathfrak{E} = \mathfrak{E}^{(11|32)}$,

$$\{Q_\alpha, Q_\beta\} = \Gamma_{\alpha\beta}^a P_a. \quad (3.5)$$

Considered as forms on rigid superspace ($\Sigma^{(D|n)}$ in general), one identifies x^a and θ^α with the coordinates $Z^M = (x^a, \theta^\alpha)$ of this superspace. Notice that the standard $D=11$ *rigid* superspace is the group manifold of the supertranslations group $\Sigma^{(11|32)}$. When $E^a = \Pi^a$ and $\psi^\alpha = \Pi^\alpha$ are forms on spacetime, x^a are still spacetime coordinates while θ^α are Grassmann functions, $\theta^\alpha = \theta^\alpha(x)$, the Volkov-Akulov Goldstone fermions [38].

Thus, when the curvatures in Eqs. (2.28), (2.29), (2.30) are set to zero, the one-form gauge fields E^a and ψ^α of supergravity become identified with the MC forms of $\mathfrak{E}^{(11|32)}$ which have a natural representation as one-forms on rigid superspace $\Sigma^{(11|32)}$. If we now look at the $D = 11$ supergravity FDA, it is also natural to ask what is the ‘flat’ limit of the three-form gauge field A_3 . To answer this question we set further $\mathbf{R}_4 = 0$ in (2.31) and find $dA_3 = w_4$ where w_4 is the ‘flat value’ of the bifermionic form a_4 in Eq. (2.46),

$$dA_3 = w_4 := -\frac{1}{4}\Pi^\alpha \wedge \Pi^\beta \wedge \Pi^a \wedge \Pi^b \Gamma_{ab\alpha\beta} . \quad (3.6)$$

The *r.h.s.* of this equation, w_4 , is a supersymmetry invariant closed four-form. It is also exact (trivial) in the de Rahm cohomology, $w_4 = dw_3(x, \theta)$, but the three-form $w_3(x, \theta)$ is not invariant under rigid supersymmetry transformations¹⁰ since $w_3(x, \theta)$ involves the Grassmann coordinates θ explicitly (not through the MC one-forms Π^a and Π^α). In fact, the superspace three-form $w_3(x, \theta)$ is well known, as its pull-back to the worldvolume \mathcal{W}^3 is the Wess–Zumino term in the $D=11$ supermembrane action [39]. Thus, $w_4 := dw_3$ (dA_3) defines as a nontrivial $\mathfrak{E}^{(11|32)}$ CE four-cocycle, since it is not the exterior derivative of an *invariant* form on $\Sigma^{(11|32)}$. To trivialize this CE cocycle *i.e.*, to write w_3 in terms of the MC forms of some superalgebra, one may, following the point of view of [40, 9], enlarge the superspace group manifold Σ (superalgebra \mathfrak{E}) to $\tilde{\Sigma}$ ($\tilde{\mathfrak{E}}$) by adding a number of additional bosonic and fermionic coordinates (generators) (see [10] and Sec. 6 for further discussion). We will show below that such extended superspaces [supergroups] $\tilde{\Sigma}$ exist, with 528 bosonic and 64 fermionic dimensions, and that they can be identified with the nontrivial deformations $\tilde{\Sigma}^{(528|32+32)}(s \neq 0)$ of the supergroup manifold $\tilde{\Sigma}^{(528|32+32)}(0)$ [10], the algebra of the latter being related to an *expansion* [13] of $osp(1|32)$.

Before turning to the trivialization of the $\Sigma^{(11|32)}$ four-cocycle on the Lie superalgebra $\tilde{\mathfrak{E}}(s) = \tilde{\mathfrak{E}}^{(528|32+32)}(s)$ and to the equivalent problem of finding the composite structure of the A_3 field of CJS supergravity in terms of $\tilde{\mathfrak{E}}(s)$ gauge fields, let us comment on the ‘flat’ limit of the dual six-form field. Setting $\mathbf{R}_7 = 0$, one finds from (2.32)

$$dA_6 = -A_3 \wedge dA_3 + \frac{i}{2 \cdot 5!} \Pi^\alpha \wedge \Pi^\beta \wedge \Pi^{a_5} \wedge \dots \wedge \Pi^{a_1} \Gamma_{a_1 \dots a_5 \alpha \beta} := -A_3 \wedge dA_3 + b_7^0 , \quad (3.7)$$

where b_7^0 is the ‘flat’ value of the bifermionic seven form b_7 , Eq. (2.47). The consistency condition $ddA_6 = 0$ is satisfied due to (3.6) and the $D=11$ identity

$$\Gamma_{a(\alpha\beta} \Gamma^{ab_1 \dots b_4}_{\gamma\delta)} = 3\Gamma^{[b_1 b_2}_{(\alpha\beta} \Gamma^{b_3 b_4]}_{\gamma\delta)} . \quad (3.8)$$

However, dA_6 is not a CE seven-cocycle on $\mathfrak{E}^{(11|32)}$ because, as stated, the three-form $A_3 = w_3(x, \theta)$ is not invariant under the standard supersymmetry group transformations. This,

¹⁰Note that, under *local* supersymmetry, $\delta_\varepsilon A_3 = -\frac{1}{2}d\theta^\alpha \wedge \Pi^a \wedge \Pi^b \Gamma_{ab\alpha\beta} \varepsilon^\beta(x) + d\alpha_2$, where we allow for the presence of an arbitrary 2-form α_2 which could be identified with the parameter of the three-form gauge transformations; for the general FDA case, where the curvatures are nonvanishing, defining $\delta_\varepsilon A_3 = -\frac{1}{2}\psi^\alpha \wedge E^a \wedge E^b \Gamma_{ab\alpha\beta} \varepsilon^\beta(x) + d\alpha_2$ one finds that \mathbf{R}_4 is invariant under local supersymmetry.

however, is the case on the $\tilde{\Sigma}^{(528|32+32)}(s \neq 0)$ superspace, where, as we show in the next section, dA_3 itself is ‘trivialized’ *i.e.* A_3 is expressed as a product of the MC one-forms invariant under the enlarged supersymmetry group $\tilde{\Sigma}^{(528|32+32)}(s \neq 0)$. It would be interesting to see whether the cocycle (3.7) is already trivial on the $\tilde{\Sigma}^{(528|32+32)}(s \neq 0)$ superspace or whether a further extension is needed. Another, equivalent, formulation of the same problem is whether the six-form A_6 may be expressed, as A_3 , in terms of the $\tilde{\Sigma}^{(528|32+32)}(s \neq 0)$ gauge fields. This corresponds to looking for an underlying gauge group structure for the selfdual formulation of $D = 11$ supergravity [41] (see also [26], [2]).

4 Trivialization of the four-cocycle and the underlying gauge group structure of D=11 supergravity

The general FDA defined by the set of Eqs. (2.28), (2.29), (2.30) may be treated as a ‘gauging’ of the super-Poincaré group described by the superPoincaré algebra MC equations (3.1), (3.2), (3.3). In $D = 4$, where the supergravity multiplet consists of the graviton and the gravitino, this allows one to state that supergravity is a gauge theory of the superPoincaré group (see references in [11] and [42]). The one-forms $E^a(x)$, $\psi^\alpha(x)$, $\omega^{ab}(x)$ are then treated as gauge fields for local translations (or general coordinate transformations), local supersymmetry and Lorentz rotations. However, for $D = 11$ CJS supergravity this is prevented by the presence of an ‘elementary’ three-form gauge field $A_3(x)$.

4.1 The $\tilde{\mathfrak{C}}^{(528|32+32)}(s)$ family of superalgebras

As stated in [5], the problem is whether the FDA (2.28), (2.29), (2.30) may be completed with a number of additional *one*-forms and their curvatures in such a way that the three-form A_3 obeying (2.31) is constructed from one-forms, becoming composite rather than fundamental or ‘elementary’. This problem is equivalent to trivializing the $\Sigma^{(11|32)}$ four-cocycle $dA_3 = w_4$, Eq. (3.6), on the algebra of an *enlarged* superspace group.

Indeed, A_3 may be constructed in terms of the graviton, gravitino, an antisymmetric second rank tensor one-form B_1^{ab} , a fifth rank antisymmetric tensor one-form $B_1^{a_1 \dots a_5}$ and an additional *fermionic spinor* one-form η_1^α ,

$$A_3 = A_3(E^a, \psi^\alpha; B_1^{ab}, B_1^{abcde}, \eta_{1\alpha}). \quad (4.1)$$

The curvatures of the new forms are defined by

$$\mathcal{B}_2^{ab} = DB_1^{ab} + \psi^\alpha \wedge \psi^\beta \Gamma_{\alpha\beta}^{ab}, \quad (4.2)$$

$$\mathcal{B}_2^{a_1 \dots a_5} = DB_1^{a_1 \dots a_5} + i\psi^\alpha \wedge \psi^\beta \Gamma_{\alpha\beta}^{a_1 \dots a_5}, \quad (4.3)$$

$$\mathcal{B}_{2\alpha} = D\eta_{1\alpha} - i\delta E^a \wedge \psi^\beta \Gamma_{a\alpha\beta} - \gamma_1 B_1^{ab} \wedge \psi^\beta \Gamma_{ab\alpha\beta} - i\gamma_2 B_1^{a_1 \dots a_5} \wedge \psi^\beta \Gamma_{a_1 \dots a_5 \alpha\beta}, \quad (4.4)$$

where D is the Lorentz covariant derivative and δ , γ_1 , γ_2 are constants to be fixed.

Let us discuss enlarging the FDA of Eqs. (2.28), (2.29) and (2.30) by the one-forms and curvatures of Eqs. (4.2), (4.3) and (4.4) in more detail. The constants in these expressions must obey one single relation (1.5) [5],

$$\delta + 10\gamma_1 - 720\gamma_2 = 0, \quad (4.5)$$

which comes from the selfconsistency (closure under the exterior derivative) of eq. (4.4) after using the rest of the FDA equations, Eqs. (2.28), (2.29), (2.30), (4.2), (4.3) and the identity

$$\Gamma_{b(\alpha\beta}\Gamma_{\gamma\delta)}^b = -\frac{1}{10}\Gamma_{ab(\alpha\beta}\Gamma_{\gamma\delta)}^{ab} = -\frac{1}{720}\Gamma_{a_1\dots a_5(\alpha\beta}\Gamma_{\gamma\delta)}^{a_1\dots a_5} . \quad (4.6)$$

Setting the curvatures in Eqs. (4.2)–(4.4) equal to zero and omitting the trivial Lorentz connection, the resulting equations plus Eqs. (3.1)–(3.3) become the MC equations of an enlarged supersymmetry algebra with the following nonvanishing (anti)commutators

$$\{Q_\alpha, Q_\beta\} = \Gamma_{\alpha\beta}^a P_a + i\Gamma_{\alpha\beta}^{ab} Z_{ab} + \Gamma_{\alpha\beta}^{a_1\dots a_5} Z_{a_1\dots a_5} , \quad (4.7)$$

$$[P_a, Q_\alpha] = \delta \Gamma_{a\alpha\beta} Q'^\beta ,$$

$$[Z_{ab}, Q_\alpha] = i\gamma_1 \Gamma_{ab\alpha\beta} Q'^\beta ,$$

$$[Z_{a_1\dots a_5}, Q_\alpha] = \gamma_2 \Gamma_{a_1\dots a_5\alpha\beta} Q'^\beta , \quad (4.8)$$

$$[Q'^\alpha, all] = 0 . \quad (4.9)$$

Clearly, as long as the constant γ_1 (or δ) is nonvanishing, it can be included in the normalization of the additional fermionic central generator Q'^α or, equivalently, in the one-form $\eta_{1\alpha}$ in (4.4).

Upon solving condition (4.5) on the constants δ , γ_1 , γ_2 in terms of one parameter s (1.6) and γ_1 , one writes the algebra (4.7), (4.8) in the form of Eq. (4.7) and

$$\begin{aligned} [P_a, Q_\alpha] &= 2\gamma_1(s+1) \Gamma_{a\alpha\beta} Q'^\beta , \\ [Z_{a_1 a_2}, Q_\alpha] &= i\gamma_1 \Gamma_{a_1 a_2 \alpha\beta} Q'^\beta , \\ [Z_{a_1\dots a_5}, Q_\alpha] &= 2\gamma_1\left(\frac{s}{6!} + \frac{1}{5!}\right) \Gamma_{a_1\dots a_5 \alpha\beta} Q'^\beta . \end{aligned} \quad (4.10)$$

Thus, one concludes that the family of the fermionic central extensions ($[Q'^\alpha, all] = 0$) of the M-theory superalgebra described by Eqs. (4.7), (4.8) is effectively one-parametric. Following [10] we denote the family of superalgebras given by Eqs. (4.7) and (4.10) by $\tilde{\mathfrak{E}}(s) = \tilde{\mathfrak{E}}^{(528|32+32)}(s)$, and by $\tilde{\Sigma}(s) = \tilde{\Sigma}^{(528|32+32)}(s)$ the associated extended superspace group manifolds. The MC equations of $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$ are given by the above FDA equations for zero curvatures *i.e.*,

$$\begin{aligned} d\Pi^a &= -i\Pi^\alpha \wedge \Pi^\beta \Gamma_{\alpha\beta}^a , \\ d\Pi^\alpha &= 0 , \end{aligned} \quad (4.11)$$

$$d\Pi^{a_1 a_2} = -\Pi^\alpha \wedge \Pi^\beta \Gamma_{\alpha\beta}^{a_1 a_2} , \quad (4.12)$$

$$d\Pi^{a_1\dots a_5} = -i\Pi^\alpha \wedge \Pi^\beta \Gamma_{\alpha\beta}^{a_1\dots a_5} , \quad (4.13)$$

$$d\Pi'_\alpha = -2\gamma_1 \Pi^\beta \wedge \left(i(s+1) \Pi^a \Gamma_a + \frac{1}{2} \Pi^{ab} \Gamma_{ab} + i\left(\frac{s}{6!} + \frac{1}{5!}\right) \Pi^{a_1\dots a_5} \Gamma_{a_1\dots a_5} \right)_{\beta\alpha} . \quad (4.14)$$

4.2 The $\tilde{\mathfrak{E}}^{(528|32+32)}(0)$ superalgebra and its associated FDA

The $D=11$ Fierz identity

$$\delta_{(\alpha} \gamma \delta_{\beta)}^\delta = \frac{1}{32} \Gamma_{\alpha\beta}^a \Gamma_a^\gamma \Gamma^\delta - \frac{1}{64} \Gamma^{a_1 a_2}_{\alpha\beta} \Gamma_{a_1 a_2}^\gamma \Gamma^\delta + \frac{1}{32 \cdot 5!} \Gamma^{a_1\dots a_5}_{\alpha\beta} \Gamma_{a_1\dots a_5}^\gamma \Gamma^\delta \quad (4.15)$$

allows one to collect the set of one-forms E^a , B_1^{ab} , B_1^{abcde} into one symmetric spin-tensor one-form $\mathcal{E}^{\alpha\beta}$,

$$\mathcal{E}^{\alpha\beta} = \frac{1}{32} \left(E^a \Gamma_a^{\alpha\beta} - \frac{i}{2} B_1^{a_1 a_2} \Gamma_{a_1 a_2}^{\alpha\beta} + \frac{1}{5!} B_1^{a_1 \dots a_5} \Gamma_{a_1 \dots a_5}^{\alpha\beta} \right). \quad (4.16)$$

The curvatures

$$\mathcal{R}^{\alpha\beta} = D\mathcal{E}^{\alpha\beta} + i\psi^\alpha \wedge \psi^\beta, \quad (4.17)$$

(2.29), (2.30) of the set of one-forms $\mathcal{E}^{\alpha\beta}$, ψ^α , ω^{ab} satisfy the Bianchi identities

$$\begin{aligned} D\mathcal{R}^{\alpha\beta} &:= D\mathcal{R}^{\alpha\beta} + 2\mathcal{E}^{\gamma(\alpha} \wedge \mathbf{R}_{\gamma}^{\beta)} - 2i\psi^{(\beta} \wedge \mathbf{R}^{\alpha)} \equiv 0, \\ \mathbf{R}_{\gamma}^{\beta} &\equiv \frac{1}{4} \mathbf{R}^{ab} \Gamma_{ab\gamma}^{\beta}, \end{aligned} \quad (4.18)$$

(2.34) and (2.35). This FDA includes the Lorentz-spin connection ω^{ab} and its curvature \mathbf{R}^{ab} , Eq. (2.30).

If we move to the flat limit where all curvatures are zero, and set to zero the spin connection, $\omega^{ab} = 0$, the one-forms $E^a = \Pi^a$, $\psi^\alpha = \Pi^\alpha$, $B_1^{ab} = \Pi^{ab}$, $B_1^{a_1 \dots a_5} = \Pi^{a_1 \dots a_5}$ obey the MC equations (4.11)–(4.13). These can be collected in the compact expression

$$d\Pi^{\alpha\beta} = -i\Pi^\alpha \wedge \Pi^\beta, \quad d\Pi^\alpha = 0 \quad (4.19)$$

clearly exhibiting a $GL(n)$ symmetry ($\Pi^{\alpha\beta}$ is the flat limit of $\mathcal{E}^{\alpha\beta}$ in (4.16)). This is the automorphism symmetry of the M-theory superalgebra $\tilde{\mathfrak{E}}^{(528|32)}$ which is defined by the MC equations (4.19) and can be obtained from any of the $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$ superalgebras just by setting the fermionic central charge Q'^α equal to zero. Clearly, none of the $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$ superalgebras possess the full $GL(n)$ automorphism symmetry; for $s \neq 0$ they only possess the Lorentz one $SO(1, 10)$. This automorphism group is enhanced to $Sp(32)$ when $s = 0$ *i.e.*, for the special values of the constants δ , γ_1 , γ_2 given by

$$\delta = 2\gamma_1, \quad \gamma_2 = \frac{2}{5!}\gamma_1 \quad \Leftrightarrow \quad s = 0. \quad (4.20)$$

Indeed, the $\tilde{\mathfrak{E}}^{(528|32+32)}(0)$ algebra MC equations, Eqs. (4.11)–(4.14), can be collected in

$$\tilde{\mathfrak{E}}^{(528|32+32)}(0) : \quad \begin{cases} d\Pi^{\alpha\beta} = -i\Pi^\alpha \wedge \Pi^\beta, \\ d\Pi^\alpha = 0, \\ d\Pi'_\alpha = i\Pi_{\alpha\beta} \wedge \Pi^\beta, \end{cases} \quad (4.21)$$

where, for definiteness, we have set the inessential constant to $\gamma_1 = 1/64$.

We can also write in this notation the MC equations of the $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$ superalgebra,

$$\begin{aligned} \tilde{\mathfrak{E}}^{(528|32+32)}(s) : \\ \begin{cases} d\Pi^{\alpha\beta} = -i\Pi^\alpha \wedge \Pi^\beta, \\ d\Pi^\alpha = 0, \\ d\Pi'_\alpha = i(\Pi_{\alpha\beta} + s/32 \Pi^a \Gamma_{a\alpha\beta} + s/32 \cdot 6! \Pi^{a_1 \dots a_5} \Gamma_{a_1 \dots a_5 \alpha\beta}) \wedge \Pi^\beta. \end{cases} \end{aligned} \quad (4.22)$$

For $s \neq 0$ the last equation in (4.22) involves explicitly the $D=11$ gamma-matrices, so that $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$ possess only $SO(1, 10)$ automorphisms.

Softening the $\tilde{\mathfrak{E}}^{(528|32+32)}(0)$ Maurer–Cartan equations by introducing nonvanishing curvatures and a nontrivial spin connection, the Lie algebra is converted into a gauge FDA generated by E^a , ψ^α , ω^{ab} , B_1^{ab} , $B_1^{a_1, \dots, a_5}$ and $\eta_{1\alpha}$ and their curvatures (Eqs. (2.28), (2.29), (2.30), (4.2), (4.3), (4.4)) which can be rewritten in terms of

$$\mathcal{E}^{\alpha\beta}, \quad \psi^\alpha, \quad \eta_{1\alpha}; \quad (4.23)$$

$$\omega^{ab}, \quad (4.24)$$

and their curvatures,

$$\mathcal{R}^{\alpha\beta} = D\mathcal{E}^{\alpha\beta} + i\psi^\alpha \wedge \psi^\beta, \quad (4.25)$$

$$\mathbf{R}^\alpha = D\psi^\alpha := d\psi^\alpha - \psi^\beta \wedge \omega_\beta^\alpha, \quad (4.26)$$

$$\mathcal{B}_{2\alpha} = D\eta_{1\alpha} - i\mathcal{E}_{\alpha\beta} \wedge \psi^\beta, \quad (4.27)$$

(setting again $\gamma_1 = 1/64$ in Eq. (4.27)) plus

$$\mathbf{R}^{ab} := R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c^b \quad (4.28)$$

obeying the Bianchi identities (4.18), (2.34),

$$D\mathcal{B}_{2\alpha} := D\mathcal{B}_{2\alpha} + i\mathcal{E}_{\alpha\beta} \wedge \mathbf{R}^\beta - i\mathcal{R}_{\alpha\beta} \wedge \psi^\beta + \mathcal{R}_\alpha^\beta \wedge \eta_{1\beta} \equiv 0, \quad (4.29)$$

and (2.35). The $\alpha\beta$ indices are raised and lowered in (4.27) and (4.29) by the charge conjugation matrix $C_{\alpha\beta}$ and its inverse $C^{\alpha\beta}$; notice that the gamma matrices now only appear in the spin connection $\omega_\alpha^\beta = 1/4\omega^{ab}\Gamma_{ab\alpha}^\beta$ that enters in the covariant derivative D . Thus replacing formally the spin connection ω_α^β by a more general symplectic one Ω_α^β (restricted only by $DC_{\alpha\beta} = 0$ which implies $\Omega_{\alpha\beta} := \Omega_\alpha^\gamma C_{\beta\gamma} = \Omega_{\beta\alpha} = \Omega_{(\alpha\beta)}$) we arrive at a FDA possessing a local $Sp(32, \mathbb{R})$ symmetry.

As discussed in [10], the superalgebra $\tilde{\mathfrak{E}}(0) \oplus so(1, 10)$, corresponding to the FDA (4.23)–(4.28) without the replacement of ω_α^β by Ω_α^β , is an expansion [13] of the supergroup $OSp(1|32)$, denoted $OSp(1|32)(2, 3, 2)$, of dimension $647 = 583 + 64$; the superalgebra $\mathfrak{E}(0) \oplus sp(32)$, corresponding to the FDA (4.23)–(4.28) with ω_α^β replaced by the $sp(32)$ valued Ω_α^β , is given by the expansion $OSp(1|32)(3, 2)$ of dimension $1120 = 1056 + 64$. We refer to [13, 10] for details.

4.3 Composite nature of the A_3 three-form gauge field

The problem [5] now is to express the form A_3 defined by the CJS FDA relations (2.31), (2.21) [with (2.28)–(2.30) (2.18)–(2.20)] in terms of the one-forms B_1^{ab} , B_1^{abcde} , $\eta_{1\alpha}$ plus the original graviton and gravitino one-forms, E^a and ψ^α . For it, we write the most general expression for a three-form A_3 in terms of the above one-forms,

$$\begin{aligned} A_3 = & \frac{\lambda}{4} B_1^{ab} \wedge E_a \wedge E_b - \frac{\alpha_1}{4} B_{1ab} \wedge B_{1c}^b \wedge B_1^{ca} - \frac{\alpha_2}{4} B_{1b_1 a_1 \dots a_4} \wedge B_{1b_2}^{b_1} \wedge B_{1b_2 a_1 \dots a_4}^{b_2 a_1 \dots a_4} - \\ & - \frac{\alpha_3}{4} \epsilon_{a_1 \dots a_5 b_1 \dots b_5 c} B_1^{a_1 \dots a_5} \wedge B_{1b_1 \dots b_5}^{b_1 \dots b_5} \wedge E^c - \\ & - \frac{\alpha_4}{4} \epsilon_{a_1 \dots a_6 b_1 \dots b_5} B_1^{a_1 a_2 a_3}{}_{c_1 c_2} \wedge B_{1b_1 \dots b_5}^{a_4 a_5 a_6 c_1 c_2} \wedge B_{1b_1 \dots b_5}^{b_1 \dots b_5} - \\ & - \frac{i}{2} \psi^\beta \wedge \eta_{1\alpha}^\alpha \wedge \left(\beta_1 E^a \Gamma_{a\alpha\beta} - i\beta_2 B_1^{ab} \Gamma_{ab\alpha\beta} + \beta_3 B_1^{abcde} \Gamma_{abcde\alpha\beta} \right), \end{aligned} \quad (4.30)$$

and look for the values of the constants $\alpha_1, \dots, \alpha_4$, β_1, \dots, β_3 and λ such that A_3 of Eq. (4.30) obeys (2.31) for arbitrary curvatures of the one-form fields. The numerical factors in the right hand side of (4.30) are introduced to make the definition of the coefficients coincide with that in [5] while keeping our notation for the FDA and supergravity constraints. The only essential difference with [5] is the inclusion of the arbitrary coefficient λ in the first term; as we show below this leads to a one-parametric family of solutions that includes the two D'Auria–Fré ones.

Factoring out the coefficients for the various independent forms one finds a system of equations given by ¹¹

$$\beta_1 + 10\beta_2 - 6!\beta_3 = 0, \quad (4.31)$$

and

$$\lambda - 2\delta\beta_1 = 1, \quad (4.32)$$

$$\lambda - 2\gamma_1\beta_1 - 2\delta\beta_2 = 0, \quad (4.33)$$

$$3\alpha_1 + 8\gamma_1\beta_2 = 0, \quad (4.34)$$

$$\alpha_2 - 10\gamma_1\beta_3 - 10\gamma_2\beta_2 = 0, \quad (4.35)$$

$$5!\alpha_3 - \delta\beta_3 - \gamma_2\beta_1 = 0, \quad (4.36)$$

$$\alpha_2 - 5!10\gamma_2\beta_3 = 0, \quad (4.37)$$

$$\alpha_3 - 2\gamma_2\beta_3 = 0, \quad (4.38)$$

$$3\alpha_4 + 10\gamma_2\beta_3 = 0. \quad (4.39)$$

Eq. (4.31) comes from the cancellation of terms proportional to the $\psi \wedge \psi \wedge \psi \wedge \eta_1$ form in Eq. (2.31) with (4.30); Eqs. (4.32)–(4.39) from the cancellation of terms proportional to $\psi \wedge \psi$ times different products of bosonic forms E^a , B_1^{ab} , B_1^{abcde} [namely, $\psi^\alpha \wedge \psi^\beta \wedge E^a \wedge E^b \Gamma_{ab\alpha\beta}$ for (4.32); $\psi^\alpha \wedge \psi^\beta \wedge B_1^{ab} \wedge E_b \Gamma_{a\alpha\beta}$ for (4.33); *etc.*].

For the nonvanishing curvatures of the one-form fields, Eq. (2.31) with (4.30) gives also the expression for the four-form curvature \mathbf{R}_4 in terms of the curvatures of the one-forms; setting $\mathbf{R}^a = 0$ (which is proper in the description of supergravity constraints as well as for the component formulation of supergravity with “supersymmetric spin connections”) one gets

$$\begin{aligned} \mathbf{R}_4 = & \frac{\lambda}{4} \mathcal{B}_2^{ab} \wedge E_a \wedge E_b - \frac{3\alpha_1}{4} \mathcal{B}_{2ab} \wedge B_{1c}^b \wedge B_1^{ca} - \frac{\alpha_2}{2} \mathcal{B}_{2a_1\dots a_5} \wedge B_{1b}^{a_1} \wedge B_1^{ba_2\dots a_5} + \\ & + \frac{\alpha_2}{4} B_{1a_1\dots a_5} \wedge \mathcal{B}_{2b}^{a_1} \wedge B_1^{ba_2\dots a_5} - \frac{\alpha_3}{2} \epsilon_{a_1\dots a_5 b_1\dots b_5 c} E^c \wedge B_{1a_1\dots a_5} \wedge \mathcal{B}_2^{b_1\dots b_5} - \\ & - \frac{\alpha_4}{4} \epsilon_{a_1\dots a_6 b_1\dots b_5} B_{1c_1 c_2}^{a_1 a_2 a_3} \wedge B_{1c_1 c_2}^{a_4 a_5 a_6} \wedge \mathcal{B}_2^{b_1\dots b_5} - \\ & - \frac{\alpha_4}{2} \epsilon_{a_1\dots a_6 b_1\dots b_5} B_{1c_1 c_2}^{a_4 a_5 a_6} \wedge B_{1b_1\dots b_5}^{a_1} \wedge \mathcal{B}_2^{a_1 a_2 a_3 c_1 c_2} - \\ & - \frac{i}{2} \psi^\beta \wedge \eta_1^\alpha \wedge \left(-i\beta_2 \mathcal{B}_2^{ab} \Gamma_{ab\alpha\beta} + \beta_3 \mathcal{B}_2^{abcde} \Gamma_{abcde\alpha\beta} \right) + \\ & + \frac{i}{2} \psi^\beta \wedge \left(\beta_1 E^a \Gamma_{a\alpha\beta} - i\beta_2 B_1^{ab} \Gamma_{ab\alpha\beta} + \beta_3 B_1^{abcde} \Gamma_{abcde\alpha\beta} \right) \wedge \mathcal{B}_2^\alpha + \\ & + \frac{i}{2} \eta^\alpha \wedge \left(\beta_1 E^a \Gamma_{a\alpha\beta} - i\beta_2 B_1^{ab} \Gamma_{ab\alpha\beta} + \beta_3 B_1^{abcde} \Gamma_{abcde\alpha\beta} \right) \wedge \mathbf{R}^\beta. \end{aligned} \quad (4.40)$$

¹¹The factor 5! in (4.36) is missing in footnote 6 in [10].

Eq. (4.40) assumes that the relations (4.31)–(4.39) among the coefficients λ, α_1, \dots are satisfied. These equations, actually necessary conditions for a composite structure of the three-form A_3 , also solve the problem of trivializing the cocycle (3.6) on a suitable flat (or rigid) enlarged superspace to which we now turn.

4.4 Trivializing the four-cocycle dA_3 , enlarged superspaces, and the fields/extended superspace variables correspondence

The trivialization of the w_4 (dA_3) standard supersymmetry algebra four-cocycle on a larger Lie superalgebra implies expressing the form A_3 obeying (3.6) in terms of MC one-forms on a larger supergroup manifold, *i.e.* on a *generalized superspace* (see [9]) with additional coordinates. The above described calculations for the case of vanishing curvatures leading to Eqs. (4.31)–(4.39) considered E^a , ψ^α , B_1^{ab} , B_1^{abcde} , $\eta_{1\alpha}$ as independent one-forms. This is tantamount to looking for a trivialization of the four-cocycle w_4 on an extended superalgebra $\tilde{\mathfrak{e}}^{(528|32+32)}$ associated to the *rigid* superspace (group manifold) $\tilde{\Sigma}^{(528|32+32)}$ with $517 = 55 + 462$ bosonic and 32 fermionic *additional* coordinates.

The original 32 fermionic coordinates θ^α are associated with the fermionic one-forms ψ^α , which by Eq. (2.29) become closed, $d\psi^\alpha = 0$, for vanishing curvature $\mathbf{R}^\alpha = 0$ and trivial spin connection $\omega^{ab} = 0$; as a closed form, $\psi^\alpha = d\theta^\alpha \equiv \Pi^\alpha$. Similarly, $\mathbf{R}^a = 0$, now reads $dE^a + id\theta^\alpha \wedge d\theta^\beta \Gamma_{\alpha\beta}^a = 0$ and has the invariant solution $E^a = \Pi^a = dx^a - id\theta^\alpha \Gamma_{\alpha\beta}^a \theta^\beta$ on the standard superspace $\Sigma^{(11|32)}$ of coordinates $Z = (x^\mu, \theta^\alpha)$. On $\Sigma^{(11|32)}$ all other differential forms, *e.g.* B_1^{ab} and $B_1^{a_1 \dots a_5}$, can be expressed (*e.g.*, $B_1^{ab} = E^d B_d^{ab}(Z) + E^\gamma B_\gamma^{ab}(Z)$) in the basis provided by $E^a = \Pi^a$ and $\psi^\alpha = \Pi^\alpha$. This is true also for the curved standard superspace, but in this case E^a and $\psi^\alpha = E^\alpha$ are ‘soft’ one-forms and not the invariant MC forms Π^a and Π^α .

If one considers the bosonic differential one-forms B_1^{ab} and $B_1^{a_1 \dots a_5}$ as independent, one implicitly assumes that $dB_1^{ab} = -d\theta^\alpha \wedge d\theta^\beta \Gamma_{\alpha\beta}^{ab}$ and $dB_1^{a_1 \dots a_5} = -id\theta^\alpha \wedge d\theta^\beta \Gamma_{\alpha\beta}^{a_1 \dots a_5}$ (see Eqs. (4.12), (4.13)) may be solved in terms of invariant one-forms

$$\begin{aligned} B_1^{ab} &= \Pi^{ab} := dy^{ab} - d\theta^\alpha \Gamma_{\alpha\beta}^{ab} \theta^\beta, \\ B_1^{a_1 \dots a_5} &= \Pi^{a_1 \dots a_5} := dy^{a_1 \dots a_5} - id\theta^\alpha \Gamma_{\alpha\beta}^{a_1 \dots a_5} \theta^\beta. \end{aligned} \quad (4.41)$$

The new parameters y^{ab} and $y^{a_1 \dots a_5}$ entering in $B_1^{a_1 a_2}$ and $B_1^{a_1 \dots a_5}$ constitute the additional $(55 + 462 = 517)$ bosonic variables of an extended superspace. This is the extended $\Sigma^{(528|32)}$ rigid superspace, which may be considered as a supergroup for which Π^a , $\Pi^{a_1 a_2}$, Π^{a_1, \dots, a_5} and Π^α are all invariant MC forms¹². When the curvatures are not zero, and in particular $\mathcal{B}_2^{ab} \neq 0$, $\mathcal{B}_2^{a_1 \dots a_5} \neq 0$ *i.e.*, the invariant one-forms $B_1^{a_1 a_2}$, $B_1^{a_1 \dots a_5}$ become ‘soft’, $\Sigma^{(528|32)}$ is non flat and no longer a group manifold.

¹²The $\Sigma^{(528|32)}$ extended superspace group may be found in our spirit by searching for a trivialization of the \mathbb{R}^{528} -valued two-cocycle $d\mathcal{E}^{\alpha\beta} = -id\theta^\alpha \wedge d\theta^\beta$, which leads to the one-form $\mathcal{E}^{\alpha\beta} = dX^{\alpha\beta} - id\theta^{(\alpha} \theta^{\beta)}$. This introduces in a natural way the 528 bosonic coordinates $X^{\alpha\beta}$ and the transformation law $\delta_\epsilon X^{\alpha\beta} = i\theta^{(\alpha} \epsilon^{\beta)}$ that makes $\mathcal{E}^{\alpha\beta}$ invariant, and hence leads to a central extension structure for the extended superspace group $\Sigma^{(528|32)}$, which is parametrized by $(X^{\alpha\beta}, \theta^\alpha)$. The 528 bosonic coordinates include, besides the standard spacetime $x^\mu = 1/32 X^{\alpha\beta} \Gamma_{\alpha\beta}^\mu$ ones, 517 = 55 + 462 tensorial additional coordinates, $y^{\mu\nu} = 1/64! X^{\alpha\beta} \Gamma_{\alpha\beta}^{\mu\nu}$ plus $y^{\mu_1 \dots \mu_5} = 1/64! X^{\alpha\beta} \Gamma_{\alpha\beta}^{\mu_1 \dots \mu_5}$. The (maximally extended in the bosonic sector) superspace $\Sigma^{(528|32)}$ transformations make of $\mathcal{E}^{\alpha\beta}$ a MC form that trivializes, on the extended superalgebra $\mathfrak{e}^{(528|32+32)}$, the non-trivial CE two-cocycle on the original odd abelian algebra $\Sigma^{(0|32)}$.

The additional fermionic form $\eta_{1\alpha}$ for the case of vanishing curvatures obeys (see Eqs. (4.14), (4.22))

$$d\eta_{1\alpha} = i \left(\delta E^a \Gamma_a - i\gamma_1 B_1^{ab} \Gamma_{ab} + \gamma_2 B_1^{a_1 \dots a_5} \Gamma_{a_1 \dots a_5} \right)_{\alpha\beta} \wedge \psi^\beta. \quad (4.42)$$

The two-form on the *r.h.s.* of this equation is a nontrivial two-cocycle on $\mathfrak{E}^{(528|32)}$. It may be trivialized by adding 32 new fermionic coordinates θ'_α that are used to solve (4.42) by

$$\begin{aligned} \eta_{1\alpha} &= \Pi'_\alpha := d\theta'_\alpha + i \left(\delta E^a \Gamma_a - i\gamma_1 B_1^{ab} \Gamma_{ab} + \gamma_2 B_1^{a_1 \dots a_5} \Gamma_{a_1 \dots a_5} \right)_{\alpha\beta} \theta^\beta \\ &- \frac{2}{3} \delta d\theta \Gamma^a \theta (\Gamma_a \theta)_\alpha + \frac{2}{3} \gamma_1 d\theta \Gamma^{ab} \theta (\Gamma_{ab} \theta)_\alpha - \frac{2}{3} \gamma_2 d\theta \Gamma^{a_1 \dots a_5} \theta (\Gamma_{a_1 \dots a_5} \theta)_\alpha. \end{aligned} \quad (4.43)$$

In the next section we will show that the trivialization of the CE cocycles encoded in the FDA of Eqs. (2.28), (2.29), (4.2)–(4.4) is possible on all the extended superalgebras $\mathfrak{E}^{(528|32+32)}(s \neq 0)$ associated with the superspace groups $\Sigma^{(528|32+32)}(s)$ parametrized by

$$\Sigma^{(528|32+32)}(s) : (x^\mu, y^{\mu\nu}, y^{\mu_1 \dots \mu_5}; \theta^\alpha, \theta'_\alpha), \quad (4.44)$$

where the θ'_α coordinate (the ‘second’ 32, corresponding to the fermionic central charge) is associated with the one-form $\eta_{1\alpha}$ through (4.43).

The softening of the $\mathfrak{E}^{(528|32+32)}$ MC equations leads to the associated gauge FDA, with as many one-form gauge fields as group parameters. This is one more example of the fields/extended superspace coordinates correspondence already mentioned (See also Sec. 6).

4.5 Underlying gauge superalgebras for $D = 11$ supergravity and their associated A_3 composite structures

As it was discussed above, the constants $\delta, \gamma_1, \gamma_2$ restricted by Eq. (4.5) or, equivalently, expressed through the parameter s by (1.6), determine the superalgebras $\tilde{\mathfrak{E}}(s) = \tilde{\mathfrak{E}}^{(528|32+32)}(s)$ that are not isomorphic.

On the other hand, these constants appear in the system of equations (4.31)–(4.39) as parameters. Clearly, as we have found nine equations for eight constants $\alpha_1, \dots, \alpha_4, \beta_1, \beta_2, \beta_3$, and λ , the existence of solutions for the system of equations (4.31)–(4.39) is not guaranteed for arbitrary values of $\delta, \gamma_1, \gamma_2$ obeying (4.5), *i.e.* for $\tilde{\mathfrak{E}}(s)$ with an arbitrary s . One might expect to have one more condition on these constants that would fix them completely up to a rescaling of the ‘new’ fermionic form $\eta_{1\alpha}$ *i.e.*, that would fix completely the parameter s in (1.6) and, hence, would select only *one* representative of the $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$ family of superalgebras. However, already the existence of *two* solutions [5] of the more restricted system (4.31)–(4.39) for $\lambda = 1$, indicates that this is not the case. Indeed, the system (4.31)–(4.39) contains dependent equations. These may be identified as the equations for the constants β_2 and β_3 that come from the two equations for α_2 , Eqs. (4.35) and (4.37),

$$\gamma_2 \beta_2 + (\gamma_1 - 5! \gamma_2) \beta_3 = 0, \quad (4.45)$$

and the equation

$$10\gamma_2 \beta_2 - (\delta + 4 \cdot 5! \gamma_2) \beta_3 = 0, \quad (4.46)$$

obtained from the two expressions with α_3 , Eqs. (4.36) and (4.38), after β_1 is removed by means of (4.31). One can easily check that Eq. (4.46) coincides with (4.45) as far as δ , γ_1 and γ_2 obey (4.5).

Thus the general solution of the system of Eqs. (4.31)–(4.39) plus (4.5) is effectively one-parametric. It may be given in terms of two parameters δ and γ_1 ,

$$\begin{aligned}
\beta_1 &= \frac{2}{5} \frac{5\gamma_1 - \delta}{(2\gamma_1 - \delta)^2}, \\
\beta_2 &= \frac{1}{10} \frac{4\gamma_1 + \delta}{(2\gamma_1 - \delta)^2}, \\
\beta_3 &= \frac{1}{10 \cdot 5!} \frac{10\gamma_1 + \delta}{(2\gamma_1 - \delta)^2}, \\
\alpha_1 &= -\frac{8}{3} \gamma_1 \beta_2 = -\frac{4}{15} \frac{\gamma_1(4\gamma_1 + \delta)}{(2\gamma_1 - \delta)^2}, \\
\alpha_2 &= 10 \cdot 5! \gamma_2 \beta_3 = \frac{1}{6!} \frac{(10\gamma_1 + \delta)^2}{(2\gamma_1 - \delta)^2}, \\
\alpha_3 &= 2\gamma_2 \beta_3 = \frac{1}{5 \cdot 6! \cdot 5!} \frac{(10\gamma_1 + \delta)^2}{(2\gamma_1 - \delta)^2}, \\
\alpha_4 &= -\frac{10}{9} \gamma_2 \beta_3 = -\frac{1}{9 \cdot 6! \cdot 5!} \frac{(10\gamma_1 + \delta)^2}{(2\gamma_1 - \delta)^2}, \\
\lambda &= 1 + 2\delta\beta_1 = \frac{1}{5} \frac{20\gamma_1^2 + \delta^2}{(2\gamma_1 - \delta)^2}.
\end{aligned} \tag{4.47}$$

However, as the value of one parameter (δ if nonvanishing, γ_1 otherwise) can be used to rescale the new fermionic form $\eta_{1\alpha}$ we see that, effectively, there is a *one-parameter family of solutions*. This indicates that the trivialization of the four-cocycle is possible on (almost all) the enlarged supergroup manifolds $\tilde{\Sigma}^{(528|32+32)}(s)$ associated with the superalgebras $\tilde{\mathfrak{C}}^{(528|32+32)}(s)$.

To find the singular points, let us write the general solution (4.47) in terms of the parameter s as defined in Eq. (1.6),

$$\begin{aligned}
&\tilde{\mathfrak{C}}^{(528|32+32)}(s) : \\
&\delta = 2\gamma_1(s+1), \quad \gamma_2 = 2\gamma_1\left(\frac{s}{6!} + \frac{1}{5!}\right); \\
&\lambda = \frac{1}{5} \frac{s^2 + 2s + 6}{s^2}, \\
&\beta_1 = -\frac{1}{10\gamma_1} \frac{2s-3}{s^2}, \quad \beta_2 = \frac{1}{20\gamma_1} \frac{s+3}{s^2}, \\
&\beta_3 = \frac{3}{10 \cdot 6! \gamma_1} \frac{s+6}{s^2}, \\
&\alpha_1 = -\frac{1}{15} \frac{2s+6}{s^2}, \quad \alpha_2 = \frac{1}{6!} \frac{(s+6)^2}{s^2}, \\
&\alpha_3 = \frac{1}{5 \cdot 6! \cdot 5!} \frac{(s+6)^2}{s^2}, \quad \alpha_4 = -\frac{1}{9 \cdot 6! \cdot 5!} \frac{(s+6)^2}{s^2}.
\end{aligned} \tag{4.48}$$

We see that the only forbidden value is $s = 0$. Thus Eqs. (4.48) or, equivalently, (4.47), trivialize the $\tilde{\mathfrak{C}}^{(11|32)}$ CE four-cocycle on the extended $\tilde{\mathfrak{C}}^{(528|32+32)}(s \neq 0)$ superalgebra, with

associated supergroup manifold $\tilde{\Sigma}^{(528|32+32)}(s \neq 0)$. The same Eqs. (4.48) (or (4.47)) determine the A_3 three-form gauge field by (4.30) in terms of the one-form gauge fields of the $\tilde{\mathfrak{E}}^{(528|32+32)}(s \neq 0)$ superalgebra; these make A_3 a composite, rather than a fundamental field. We stress once more that the values of δ , γ_1 and γ_2 determine the Lie superalgebras $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$ associated with $\Sigma^{(528|32+32)}(s)$, while those of $\alpha_1, \dots, \alpha_4$ and β_1, \dots, β_3 determine the expression of A_3 in Eq. (4.30) (the trivialization of the cocycle).

Setting $\lambda = 1$, as in [5], we find only *two* solutions, one corresponding to $s = 3/2$

$$\begin{aligned} \tilde{\mathfrak{E}}^{(528|32+32)}(3/2) : \\ \delta = 5\gamma_1 \neq 0, \quad \gamma_2 = \frac{\gamma_1}{2 \cdot 4!}; \\ \lambda = 1, \\ \beta_1 = 0, \quad \beta_2 = \frac{1}{10\gamma_1}, \quad \beta_3 = \frac{1}{6! \gamma_1}, \\ \alpha_1 = -\frac{4}{15}, \quad \alpha_2 = \frac{25}{6!}, \quad \alpha_3 = \frac{1}{6! \cdot 4!}, \quad \alpha_4 = -\frac{1}{54(4!)^2}, \end{aligned} \quad (4.49)$$

and a second corresponding to $s = -1$,

$$\begin{aligned} \tilde{\mathfrak{E}}^{(528|32+32)}(-1) : \\ \delta = 0, \quad \gamma_2 = \frac{1}{3 \cdot 4!}, \quad \gamma_1 \neq 0; \\ \lambda = 1, \\ \beta_1 = \frac{1}{2\gamma_1}, \quad \beta_2 = \frac{1}{10\gamma_1}, \quad \beta_3 = \frac{1}{4 \cdot 5! \gamma_1}, \\ \alpha_1 = -\frac{4}{15}, \\ \alpha_2 = \frac{25}{6!}, \quad \alpha_3 = \frac{1}{6! \cdot 4!}, \quad \alpha_4 = -\frac{1}{54(4!)^2}. \end{aligned} \quad (4.50)$$

Notice that the values of $\alpha_{1,2,3,4}$ are the same for both of them. The two D'Auria and Fré solutions correspond to $\delta = 1$ in Eq. (4.49) and $\gamma_1 = -1/2$ in Eq. (4.50).

Due to the presence of λ , we have more possibilities. A particularly interesting solution of our system is found by setting $s = -6$ in (4.48) or $\gamma_2 = \frac{1}{6!}(10\gamma_1 + \delta) = 0$ in (4.47):

$$\begin{aligned} \tilde{\mathfrak{E}}^{(528|32+32)}(-6) : \\ \delta = -10\gamma_1, \quad \gamma_2 = 0; \\ \lambda = \frac{1}{6}, \\ \beta_1 = \frac{1}{4! \gamma_1}, \quad \beta_2 = -\frac{1}{2 \cdot 5! \gamma_1}, \quad \beta_3 = 0, \\ \alpha_1 = \frac{1}{90}, \quad \alpha_2 = 0, \quad \alpha_3 = 0, \quad \alpha_4 = 0. \end{aligned} \quad (4.51)$$

This corresponds to an especially simple expression for the composite A_3 form in terms of the gauge fields of this $\tilde{\mathfrak{E}}^{(528|32+32)}(-6)$ superalgebra,

$$A_3 = \frac{1}{4!} B_1^{ab} \wedge E_a \wedge E_b - \frac{1}{3 \cdot 5!} B_{1ab} \wedge B_{1c}^a \wedge B_1^{cb} - \frac{i}{5! \cdot 4\gamma_1} \psi^\beta \wedge \eta^\alpha \wedge \left(10 E^a \Gamma_{a\alpha\beta} + i B_1^{ab} \Gamma_{ab\alpha\beta} \right) \quad (4.52)$$

and to a shorter version of the additional fermionic curvature (4.4)

$$\mathcal{B}_{2\alpha} = D\eta_{1\alpha} + i\gamma_1(10E^a\Gamma_a + iB_1^{ab}\Gamma_{ab})_{\alpha\beta} \wedge \psi^\beta. \quad (4.53)$$

The $\gamma_2 = 0$ choice of Eq. (4.51) implies that the underlying Lie superalgebra $\tilde{\mathfrak{E}}^{(528|32+32)}(-6)$ includes $Z_{a_1 \dots a_5}$ also as central generator (*cf.* last line in Eq. (4.8)),

$$\begin{aligned} \tilde{\mathfrak{E}}^{(528|32+32)}(-6) : \\ \{Q_\alpha, Q_\beta\} &= P_a \Gamma_{\alpha\beta}^a + Z_{ab} i \Gamma_{\alpha\beta}^{ab} + Z_{a_1 \dots a_5} \Gamma_{\alpha\beta}^{a_1 \dots a_5} , \\ [P_a, Q_\alpha] &= -10 \gamma_1 \Gamma_{a \alpha\beta} Q'^\beta , \\ [Z_{a_1 a_2}, Q_\alpha] &= i \gamma_1 \Gamma_{a_1 a_2 \alpha\beta} Q'^\beta , \\ [Z_{a_1 \dots a_5}, all] &= 0 , \\ [Q'^\alpha, all] &= 0 . \end{aligned} \quad (4.54)$$

Thus one can consistently truncate $\tilde{\mathfrak{E}}^{(528|32+32)}(-6)$ by setting the central generator $Z_{a_1 \dots a_5}$ equal to zero. In such a way one arrives at the $\tilde{\mathfrak{E}}_{min} = \tilde{\mathfrak{E}}^{(66|32+32)}$ superalgebra, whose extension by $Z_{a_1 \dots a_5}$ gives $\tilde{\mathfrak{E}}^{(528|32+32)}(-6)$ in Eq. (4.54). Explicitly, $\tilde{\mathfrak{E}}_{min}$ is the $(66 + 64)$ -dimensional superalgebra

$$\begin{aligned} \tilde{\mathfrak{E}}_{min} = \tilde{\mathfrak{E}}^{(66|32+32)} : \\ \{Q_\alpha, Q_\beta\} &= \Gamma_{\alpha\beta}^a P_a + i \Gamma_{\alpha\beta}^{a_1 a_2} Z_{a_1 a_2} , \\ [P_a, Q_\alpha] &= -10 \gamma_1 \Gamma_{a \alpha\beta} Q'^\beta , \\ [Z_{a_1 a_2}, Q_\alpha] &= i \gamma_1 \Gamma_{a_1 a_2 \alpha\beta} Q'^\beta , \\ [Q'^\alpha, all] &= 0 . \end{aligned} \quad (4.55)$$

that corresponds to the most economic enlargement of the standard supersymmetry algebra $\tilde{\mathfrak{E}}^{(11|32)}$ for which the w_4 four-cocycle (corresponding to dA_3 , Eq. (3.6)) becomes trivial. Obviously, the gauge FDA associated to $\tilde{\mathfrak{E}}_{min} = \tilde{\mathfrak{E}}^{(66|32+32)}$ does not involve $B_1^{a_1 \dots a_5}$ (see next Section).

The superalgebras $\tilde{\mathfrak{E}}^{(528|32+32)}(s \neq 0)$ account for an *underlying gauge symmetry of the $D = 11$ supergravity* in the sense that such a symmetry is hidden in the original CJS formulation and only becomes explicit when A_3 is written in terms of the one-form gauge fields of $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$, $s \neq 0$. These superalgebras may be considered themselves as nontrivial deformations of $\tilde{\mathfrak{E}}^{(528|32+32)}(0)$.

The $s = 0$ value (Eq. (4.20)) corresponds to the Lie superalgebra $\tilde{\mathfrak{E}}^{(528|32+32)}(0)$ associated with the superspace group $\tilde{\Sigma}^{(528|32+32)}(0)$. This possesses $Sp(32)$ as its automorphism group, which is not allowed when $s \neq 0$, Eqs. (4.48) or (4.47). The full $\tilde{\Sigma}^{(528|32+32)}(0) \rtimes Sp(32)$ group is isomorphic [10] to the expansion [13] $OSp(1|32)(2, 3)$ of the $OSp(1|32)$ supergroup. If the Lorentz connection is taken into account, the complete symmetry group reduces to $\tilde{\Sigma}^{(528|32+32)}(0) \rtimes SO(1, 10)$ which is isomorphic to the expansion $OSp(1|32)(2, 3, 2)$. However, $\tilde{\Sigma}^{(528|32+32)}(0)$ does not allow for a trivialization of the w_4 cocycle. Equivalently, the problem of the composite structure of A_3 form does not have a solution in terms of $\tilde{\mathfrak{E}}^{(528|32+32)}(0)$ gauge fields. This implies that the four-cocycle w_4 in Eq. (3.6) (with Lorentz rather than $Sp(32)$ invariance) may be trivialized only on the superalgebras $\tilde{\mathfrak{E}}^{(528|32+32)}(s \neq 0)$ with $SO(1, 10)$ automorphism group; the superspace $\Sigma^{(528|32+32)}(0)$, with $Sp(32)$ automorphisms, on which the new fermionic Cartan form could be given by $\eta_{1\alpha} = d\theta'_\alpha + i \Pi_{\alpha\beta} \theta^\beta - \frac{2}{3} d\theta_{(\alpha} \theta_{\beta)} \theta^\beta$ (*cf.* (4.43) for (4.20), see (4.16)), does not allow for such a trivialization. This result is in a way natural: if there should be a singularity in the solution of the cocycle trivialization conditions, it should be associated with an algebra having particular properties. This is the case for that determined by Eq. (4.20) since only for these values of the constants the rigid superspace group $\Sigma^{(528|32+32)}(0)$ automorphism symmetry is enhanced from $SO(1, 10)$ to $Sp(32)$.

4.6 An economic underlying gauge group structure for $D = 11$ supergravity and the generalized superspace $\Sigma^{(66|64)}$

Our analysis has shown that the minimal FDA allowing for a composite structure of the CJS 3-form A_3 can be defined, fixing $\gamma_1 = -1$, by

$$\mathbf{R}^a = DE^a + i\psi^\alpha \wedge \psi^\beta \Gamma_{\alpha\beta}^a, \quad (4.56)$$

$$\mathbf{R}^\alpha := D\psi^\alpha := d\psi^\alpha - \psi^\beta \wedge \omega_\beta^\alpha. \quad (4.57)$$

$$\mathbf{R}^{ab} := R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c^b, \quad (4.58)$$

$$\mathcal{B}_2^{ab} = DB_1^{ab} + \psi^\alpha \wedge \psi^\beta \Gamma_{\alpha\beta}^{ab}, \quad (4.59)$$

$$\mathcal{B}_{2\alpha} = D\eta_{1\alpha} - i(10E^a\Gamma_a + iB_1^{ab}\Gamma_{ab})_{\alpha\beta} \wedge \psi^\beta. \quad (4.60)$$

The expression for A_3 then reads

$$A_3 = \frac{1}{4!} B_1^{ab} \wedge E_a \wedge E_b - \frac{1}{3 \cdot 5!} B_{1ab} \wedge B_{1c}^a \wedge B_1^{cb} + \frac{i}{5!4} \psi^\beta \wedge \eta_1^\alpha \wedge \left(10 E^a \Gamma_{a\alpha\beta} + i B_1^{ab} \Gamma_{ab\alpha\beta} \right). \quad (4.61)$$

Its curvature is expressed by (as above, we set $\mathbf{R}^a = 0$, which is valid both for a description in standard superspace and for the component approach)

$$\begin{aligned} \mathbf{R}_4 = & \frac{1}{4!} \mathcal{B}_2^{ab} \wedge E_a \wedge E_b + \frac{1}{5!} \mathcal{B}_2^{ab} \wedge B_{1bc} \wedge B_{1a}^c + \frac{i}{5!4} \mathbf{R}^\beta \wedge \eta_1^\alpha \wedge \left(10 E^a \Gamma_{a\alpha\beta} + i B_1^{ab} \Gamma_{ab\alpha\beta} \right) - \\ & - \frac{i}{5!4} \psi^\beta \wedge \mathcal{B}_2^\alpha \wedge \left(10 E^a \Gamma_{a\alpha\beta} + i B_1^{ab} \Gamma_{ab\alpha\beta} \right) - \frac{1}{5!4} \psi^\beta \wedge \eta_1^\alpha \wedge \mathcal{B}_2^{ab} \Gamma_{ab\alpha\beta}. \end{aligned} \quad (4.62)$$

These relatively simple expressions for A_3 and \mathbf{R}_4 will be useful, in particular, for an analysis of the supergravity action with composite A_3 in Sec. 5.

For vanishing curvatures Eqs. (4.56)–(4.60) become the MC equations for the generalized supersymmetry algebra $\tilde{\mathfrak{E}}_{min} = \tilde{\mathfrak{E}}^{(66|64)}$ associated to the superspace group $\Sigma^{(66|64)}$ with coordinates

$$\tilde{\Sigma}_{min} = \tilde{\Sigma}^{(66|64)} : \quad (x^\mu, y^{\mu\nu}, \theta^\alpha, \theta'_\alpha). \quad (4.63)$$

Explicitly, these MC forms are ($\omega^{ab} = 0$)

$$E^a := \Pi^\mu \delta_\mu^a = dx^\mu \delta_\mu^a - i d\theta^\alpha \Gamma_{\alpha\beta}^a \theta^\beta, \quad (4.64)$$

$$\psi^\alpha := \Pi^\alpha = d\theta^\alpha, \quad (4.65)$$

$$B_1^{ab} := \Pi^{ab} = dy^{\mu\nu} \delta_\mu^a \delta_\nu^b - d\theta^\alpha \Gamma_{\alpha\beta}^{ab} \theta^\beta, \quad (4.66)$$

$$\eta_{1\alpha} := \Pi'_\alpha = d\theta'_\alpha + i \left(10 \Pi^a \Gamma_a + i \Pi^{ab} \Gamma_{ab} \right)_{\alpha\beta} \theta^\beta - \frac{20}{3} d\theta^\alpha \theta (\Gamma_a \theta)_\alpha - \frac{2}{3} d\theta \Gamma^{ab} \theta (\Gamma_{ab} \theta)_\alpha. \quad (4.67)$$

5 On the $D = 11$ supergravity action with composite A_3

To analyze possible dynamical consequences of a composite structure for A_3 let us follow the proposal in [5] and consider the first order supergravity action with a composite A_3 .

5.1 Equations of motion, Noether identities and extra gauge symmetries.

The equations of motion for the ‘free’ standard CJS supergravity (see Sec. 2.4 for a brief review) include Eq. (2.54),

$$\frac{\delta S}{\delta A_3} := \mathcal{G}_8 = 0. \quad (5.1)$$

We ask ourselves what would be the consequences of a composite structure of A_3 field.

Our minimal solution given by the Eqs. (4.61), (4.62), (4.60) (see (4.51)) allows for a simple discussion of this problem; this will also exhibit some properties relevant for the generic solution, Eqs. (4.30), (4.48).

To this aim one may just insert the expression (4.61), (4.62) (or (4.30) with any allowed set of constants (4.48)) into the first order action (2.44), without assuming the FDA relations (4.56)–(4.60) [or (2.28)–(2.30), (4.2)–(4.4) with (1.6)] from the start; their rôle and re-appearance will be discussed below.

As noticed in [5], the action with such an insertion would be very large and hard to handle. To overcome this difficulty we shall deal with the standard supergravity action, but with the understanding that A_3 is made out of the new and old fields and given by Eq. (4.61) or by Eqs. (4.30), (4.48), so that its variation is not independent.

Let us begin by the minimal solution. As it follows from Eq. (4.61),

$$\begin{aligned} \delta A_3 &= \frac{1}{4!} E_a \wedge E_b \wedge \delta B_1^{ab} - \frac{1}{5!} B_{1a}{}^c \wedge B_{1cb} \wedge \delta B_1^{ab} - \frac{1}{5!4} \psi^\beta \wedge \eta_1^\alpha \Gamma_{ab\beta\alpha} \wedge \delta B_1^{ab} \\ &- \frac{i}{5!4} \psi^\beta \wedge (10 E^a \Gamma_{a\alpha\beta} + i B_1^{ab} \Gamma_{ab\alpha\beta}) \wedge \delta \eta_1^\alpha, \end{aligned} \quad (5.2)$$

where we have neglected the terms with the variation of the graviton and the gravitino (which would give contributions proportional to \mathcal{G}_8 in the Einstein and Rarita-Schwinger equations of supergravity).

The variation of the supergravity action S with respect to B_1^{ab} thus reads

$$\frac{\delta S}{\delta B_{1ab}} = \frac{\delta S}{\delta A_3} \wedge \frac{\delta A_3}{\delta B_{1ab}} = \frac{1}{4!} \mathcal{G}_8 \wedge \left(E^a \wedge E^b - \frac{1}{5} B_1^{ac} \wedge B_{1c}{}^b - \frac{1}{20} \psi \wedge \eta \Gamma_{ab} \right). \quad (5.3)$$

The first order action S from [5] is the integral over $D=11$ spacetime M^{11} or an eleven-dimensional bosonic surface \mathcal{M}^{11} in the standard superspace (or even in a group manifold [5, 15]). For simplicity we will consider first in this section the M^{11} case (Eq. (2.44)); the case of the rheonomic action (Eq. (2.66)) will be considered in Sec. 5.3. The vielbein forms E^a provide a basis to express forms on M^{11} . This implies that Eq. (5.3) has the expression

$$\frac{\delta S}{\delta B_{1ab}} = \frac{1}{4!} \mathcal{G}_8 \wedge E^c \wedge E^d \mathcal{K}_{cd}{}^{ab}, \quad (5.4)$$

where the matrix

$$\mathcal{K}_{cd}{}^{ab} = \delta_{[c}{}^a \delta_{d]}^b + \frac{1}{5} B_{[c}{}^{ae} B_{d]}{}^{bf} \eta_{ef} - \frac{1}{20} \psi_{[c}{}^\beta \eta_{d]}{}^\alpha \Gamma_{\alpha\beta}{}^{ab} \quad (5.5)$$

may be quite generally assumed to be invertible. Indeed, one may think of *e.g.*, weak B_c^{ab} fields, in which case the second term is small and the third nilpotent. Then one may state that

$$\begin{aligned} \det(\mathcal{K}_{ab}{}^{cd}) \neq 0 \\ \frac{\delta S}{\delta B_1^{ab}} = 0 \quad \implies \quad \mathcal{G}_8 \wedge E^c \wedge E^d = 0 . \end{aligned} \quad (5.6)$$

The last equation clearly implies the standard equations of motion, Eq. (5.1), but now for a composite, rather than fundamental A_3 . Thus one may state, at least within the $\det(\mathcal{K}_{ab}{}^{cd}) \neq 0$ assumption, that the variation with respect to the B_1^{ab} field produces the same equations as the variation with respect to the CJS three-form A_3 ,

$$\begin{aligned} \det(\mathcal{K}_{ab}{}^{cd}) \neq 0 \\ \frac{\delta S}{\delta B_1^{ab}} = 0 \quad \implies \quad \mathcal{G}_8 := \frac{\delta S}{\delta A_3} = 0 . \end{aligned} \quad (5.7)$$

A few remarks are appropriate at this stage. The first is that the above considerations, simplified by the use of the minimal solution in Eq. (4.61), can be extended to the general case, which has a more complicated expression for A_3 that includes the $B_1^{a_1 \dots a_5}$ field. The second one is that, considered on the $D=11$ spacetime, $B_1^{ab} = E^c B_c^{ab}(x)$ involves a three index tensor $B_c^{ab}(x) = -B_c^{ba}(x)$ with reducible symmetry properties (product of two Young tableaux),

$$B_{c \ ab} \sim \square \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \end{array} , \quad (5.8)$$

and thus carries more degrees of freedom than $A_3 = 1/3! E^c \wedge E^b \wedge E^a A_{abc}(x)$ does since $A_{abc}(x)$ is fully antisymmetric $A_{abc} = A_{[abc]}$,

$$A_{abc} \sim \begin{array}{|c|} \hline \square \\ \hline \end{array} . \quad (5.9)$$

Then, as a variation with respect to B_1^{ab} produces (for $\det(\mathcal{K}_{[ab]}{}^{[cd]}) \neq 0$) the same equations as the variation with respect to A_3 , one concludes that the action for a composite A_3 must possess local symmetries that make the *extra* (*i.e.* $\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$ but *not* $\begin{array}{|c|} \hline \square \\ \hline \end{array}$) degrees of freedom in B_1^{ab} pure gauge. Similarly, one may expect to have an extra local fermionic symmetry under which the new fermionic fields $\eta_{a\alpha}(x)$ in $\eta_{1\alpha} = E^a \eta_{a\alpha}(x)$ are also pure gauge. In the case of a more general solution and accordingly a more complicated expression for A_3 , one also expects a gauge symmetry that makes the five-index one-form fields in $B_1^{a_1 \dots a_5} = E^b B_b^{a_1 \dots a_5}(x)$ pure gauge.

This is indeed the case. Actually the fact that the above $\delta B_1^{ab} = E^c \delta B_c^{ab}$ variation produces the same result as the variation with respect to $\delta A_{abc} = \delta A_{[abc]}$ (see Eqs. (5.6) and (5.1)) plays the rôle of Noether identities for all these ‘extra’ gauge symmetries. Let us show, for instance, that the supergravity action with A_3 with the simple composite structure of

Eq. (4.61) does possess extra fermionic gauge symmetries with a spinorial one-form parameter. Indeed, the equations of motion for $\eta_{1\alpha}$,

$$\frac{\delta S}{\delta \eta_1^\alpha} = 0 \quad \Rightarrow \quad \mathcal{G}_8 \wedge \psi^\beta \wedge \left(10 E^a \Gamma_{a\alpha\beta} + i B_1^{ab} \Gamma_{ab\alpha\beta} \right) = 0, \quad (5.10)$$

are satisfied identically on the B_1^{ab} equations of motion ($\mathcal{G}_8 = 0$ for $\det(\mathcal{K}_{[ab]}^{[cd]}) \neq 0$, Eqs. (5.6)). This is a Noether identity that indicates the presence of a local fermionic symmetry with parameter $\chi_{1\alpha}$, $\chi_{1\alpha} = E^a \chi_{a\alpha}$, such that

$$\delta_\chi \eta_{1\alpha} = \chi_{1\alpha}, \quad (5.11)$$

$$\delta_\chi B_1^{ab} = \frac{i}{16} \mathcal{K}^{-1[ab][cd]} \psi_c^\alpha (10 \Gamma_d + i B_d^{ef} \Gamma_{ef})_{\alpha\beta} \chi_1^\beta. \quad (5.12)$$

We can see that the transformations (5.11), (5.12) leave invariant the composite three-form A_3 (4.61) considered as a form on spacetime. If the form A_3 in (4.61) is now considered as defined on standard superspace Σ or on a larger supermanifold $\tilde{\Sigma}$, the A_3 on $\mathcal{M}^{11} \subset \Sigma$ or $\tilde{\Sigma}$ is still preserved by (5.11) for $\chi_{a\alpha}(\tilde{Z}(x))$ with a more complicated expression for $\delta_\chi B_1^{ab}(\tilde{Z}(x))$.

In the same way, having in mind that the contribution of *any* variation of the fundamental fields in δA_3 on M^{11} is always given by an *antisymmetric* third rank tensor contribution, one concludes that *any* contribution to δA_3 from an arbitrary variation of the $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ irreducible part of δB_c^{ab} (which is also an antisymmetric *contribution*) can always be compensated by a

contribution of a proper transformation of its completely antisymmetric part $\delta B_{[cba]}, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$.

When the more general form for A_3 , (Eqs. (4.30), (4.47)) is considered, the same reasoning shows that any transformations of the new form $B_1^{a_1 \dots a_5}$ can be compensated by some properly chosen B_1^{ab} transformations. The key point is that the coefficient λ in (4.47) never vanishes. Hence (omitting δE^a and $\delta \psi^\alpha$)

$$\delta A_3 = -\frac{\lambda}{4} E^c \wedge E^d \wedge \mathcal{K}_{cd}^{ab} \delta B_{1ab} + \mathcal{S}_{2a_1 \dots a_5} \wedge \delta B_1^{a_1 \dots a_5} + \mathcal{S}_2^\alpha \wedge \delta \eta_{1\alpha} = \quad (5.13)$$

$$= -\frac{\lambda}{4} E^a \wedge E^b \wedge E^c \delta B_{[c ab]} + \mathcal{O}(B_1 \wedge B_1) + \mathcal{O}(\psi_1 \wedge \eta_1), \quad (5.14)$$

$$\mathcal{K}_{cd}^{ab} = \delta_{[c}^a \delta_{d]}^b + \mathcal{O}(B_1 \wedge B_1) + \mathcal{O}(\psi_1 \wedge \eta_1), \quad (5.15)$$

$$\lambda = \frac{(20\gamma_1^2 + \delta^2)}{5(2\gamma_1 - \delta)^2} \equiv \frac{1}{5} \frac{s^2 + 2s + 6}{s^2} \neq 0$$

and the variation of the completely antisymmetric part $B_{[abc]}^{ab}$ of $B_1^{ab} = E^c B_c^{ab}$ always reproduces (for an invertible \mathcal{K} (5.15)) the same equation $\mathcal{G}_8 = 0$ as it would an independent, fundamental three-form A_3 .

5.2 Free differential algebra for the ‘new’ fields

One may ask at what stage the FDA relations (4.2), (4.3), (4.4) appear when the first order supergravity action [5, 16] with a composite A_3 field [Eqs. (4.30), (4.47)] is considered and whether there are any conditions on the new curvatures, as Eqs. (2.38), (2.39) for \mathbf{R}^a (Eq.

(2.28)) and \mathbf{R}_4 (Eq.(2.31)). Let us recall that in the first order action the latter equations are not imposed by hand, but appear as equations of motion. The action (2.44) of refs. [5, 16] include the auxiliary field F_{abcd} and the variation with respect to it produces Eq. (2.57), which is equivalent to the FDA relation (2.31) with (2.39). The variation with respect to an independent spin connection ω^{ab} produces Eq. (2.58) which is equivalent to the FDA relation (2.28) with (2.38).

As it was shown in Sec. 4, Eqs. (2.31) can be solved by expressing A_3 in terms of the one-forms E^a , B_1^{ab} , $B_1^{a_1 \dots a_5}$, ψ^α , $\eta_{1\alpha}$ by Eq. (4.30) for any set of constants given by Eqs. (4.47) (or (4.48)), provided the one-forms satisfy the FDA (2.28), (2.29), (2.30) and (4.2), (4.3), (4.4) with the same constants $\delta, \gamma_1, \gamma_2$ (with the same s). Eq. (4.30) implies also the expression for \mathbf{R}_4 through the field strengths of the one-form fields (two-form curvatures of the soft FDA algebra) (see Eq. (4.40)).

With this in mind, studying the first order supergravity action which produces Eqs. (2.57) and (2.58), one may just use the FDA relations (4.2), (4.3), (4.4) (with $\delta, \gamma_1, \gamma_2$ expressed through the same parameter s as in (4.48) used to construct A_3) to substitute the expressions

$$DB_1^{ab} = -\psi^\alpha \wedge \psi^\beta \Gamma_{\alpha\beta}^{ab} + \mathcal{B}_2^{ab}, \quad (5.16)$$

$$DB_1^{a_1 \dots a_5} = -i\psi^\alpha \wedge \psi^\beta \Gamma_{\alpha\beta}^{a_1 \dots a_5} + \mathcal{B}_2^{a_1 \dots a_5}, \quad (5.17)$$

$$D\eta_{1\alpha} = +i\delta E^a \wedge \psi^\beta \Gamma_{a\alpha\beta} + \gamma_1 B_1^{ab} \wedge \psi^\beta \Gamma_{ab\alpha\beta} + i\gamma_2 B_1^{a_1 \dots a_5} \wedge \psi^\beta \Gamma_{a_1 \dots a_5\alpha\beta} + \mathcal{B}_{2\alpha}, \quad (5.18)$$

for DB_1^{ab} , $DB_1^{a_1 \dots a_5}$, $D\eta_{1\alpha}$ in dA_3 of (2.57). What one gains making just these substitutions in Eq. (2.57) is that all the terms without curvatures coming from the first term ($dA_3(B_1^{ab}, \dots)$) are cancelled by the second term (a_4 of Eq. (2.46)). Thus, the only consequences of Eq. (2.57) would be an expression for F_4 through the newly defined field curvatures and \mathbf{R}^α (Eq. (2.29))

$$\begin{aligned} E^a \wedge E^b \wedge E^c \wedge E^d F_{abcd} &= \lambda \mathcal{B}_2^{ab} \wedge E_a \wedge E_b + \dots - \\ &\quad - 2i\psi^\beta \wedge \eta_{1\alpha} \wedge \left(-i\beta_2 \mathcal{B}_2^{ab} \Gamma_{ab} + \beta_3 \mathcal{B}_2^{abcde} \Gamma_{abcde} \right)_{\alpha\beta} \end{aligned} \quad (5.19)$$

(see Eq.(4.40) for the full expression of the right-hand-side).

As we discussed in Sec. 5.1., the variation of the new fields produces the only nontrivial equation, Eq. (5.7), which formally coincides with the original three-form equation (5.1), but now involving the composite A_3 and its field strength F_{abcd} [see Eq. (2.57) which appears as a result of varying F_{abcd} in the action (2.44), (2.45), independently of whether A_3 is composite or fundamental]. This reflects the existence of the extra gauge symmetries (see Sec. 5.1) that make that the theory with a composite A_3 carries the same number of degrees of freedom as the original CJS supergravity with a fundamental A_3 . What we have found now is that, besides of this, Eq. (5.19) is the only relation imposed on the new field strengths \mathcal{B}_2^{ab} , $\mathcal{B}_2^{a_1 \dots a_5}$, $\mathcal{B}_{2\alpha}$ by the first-order $D = 11$ supergravity action (2.44), (2.45) with a composite A_3 . This also reflects the existence of the extra gauge symmetries, as these make the detailed properties of the curvatures (\mathcal{B}_2^{abcde} , $\mathcal{B}_2^{a_1 \dots a_5}$, $\mathcal{B}_{2\alpha}$) of the additional gauge fields inessential; their only relevant properties are that the field strength F_{abcd} is constructed out of them in accordance with Eq. (5.19) and that such a composite field strength obeys Eq. (5.1).

Thus, on the one hand, the underlying gauge group structure implied by the new one-form fields allows us to treat the $D = 11$ supergravity as a gauge theory of the $\tilde{\Sigma}(s \neq 0) \bowtie SO(1, 10)$

supergroup, that replaces the superPoincaré one. On the other hand, the supergravity action (2.44), (2.45) with a composite A_3 also possesses ‘extra’ gauge symmetries (not in $\tilde{\Sigma}(s \neq 0) \rtimes SO(1, 10)$) that the *additional* degrees of freedom in the ‘new’ fields B_1^{ab} , $B_1^{a_1 \dots a_5}$, $\eta_{1\alpha}$ pure gauge (*i.e.* B_1^{ab} , $B_1^{a_1 \dots a_5}$, $\eta_{1\alpha}$ carry the same number of physical degrees of freedom as the fundamental A_3 field). In this sense the geometric $\tilde{\Sigma}(s \neq 0) \rtimes SO(1, 10)$ symmetry, although manifest, gives only a part of the gauge symmetries of the supergravity action (2.44), (2.45) with a composite A_3 .

One might conjecture that the superfluous degrees of freedom in the ‘new’ one-form fields, which are pure gauge in the pure supergravity action, would become ‘alive’ when supergravity is coupled to some M-theory objects. These could not be the usual M-branes as they couple to the standard fields and, hence, all the gauge symmetries preserving the composite A_3 would remain preserved. Thus one might think of coupling of supergravity through some new action containing explicitly the new one-forms. A guide in the search for such an action would be the preservation of the gauge symmetries of the underlying $\tilde{\Sigma}(s \neq 0) \rtimes SO(1, 10)$ gauge supergroup.

5.3 Composite A_3 in the rheonomic action. A possible way to enlarged superspace

All the above discussion on the ‘extra’ gauge symmetries (Sec. 5.1) applies also for the rheonomic action (2.66) with \mathcal{M}^{11} being an arbitrary surface in superspace. In short, this follows from the fact that all the one-forms on such a surface can be decomposed using the basis provided by the pull-back $E^a(\tilde{Z}(x)) = d\tilde{Z}^M(x)E_M^a(\tilde{Z}(x)) = dx^\mu \partial_\mu \tilde{Z}^M(x)E_M^a(\tilde{Z}(x))$ of the bosonic supervielbein $E^a(Z) = dZ^M E_M^a(Z)$.

Indeed, in the matrix $\partial_\mu \tilde{Z}^M(x)E_M^a(\tilde{Z}(x)) = E_\mu^a(\tilde{Z}(x)) + \partial_\mu \tilde{\theta}^{\tilde{\alpha}}(x)E_{\tilde{\alpha}}^a(\tilde{Z}(x))$ the first term is given by an invertible matrix E_μ^a while the second is nilpotent. Hence there exists a matrix $E_a^\mu = E_a^\mu(\tilde{Z}, \partial_\nu \tilde{Z})$ such that $E_a^\mu[E_\mu^b(\tilde{Z}(x)) + \partial_\mu \tilde{\theta}^{\tilde{\alpha}}(x)E_{\tilde{\alpha}}^b(\tilde{Z}(x))] = \delta_a^b$. This is tantamount to saying that $dx^\mu = E^a(\tilde{Z}(x))E_a^\mu$. Using this we may express a superspace differential form on \mathcal{M}^{11} in the $E^a(\tilde{Z}(x))$ basis. In particular, the superform $B_1^{ab}(Z) = dZ^M B_M^{ab}(Z)$ on \mathcal{M}^{11} , $B_1^{ab}(\tilde{Z}(x)) = d\tilde{Z}^M(x)B_M^{ab}(\tilde{Z}(x)) = dx^\mu \partial_\mu \tilde{Z}^M(x)B_M^{ab}(\tilde{Z}(x))$, may be written as $B_1^{ab}(\tilde{Z}(x)) = E^c(\tilde{Z}(x))\tilde{B}_c^{ab}$. With this in mind the above considerations on local symmetries may be extended to the case of superforms on arbitrary eleven-dimensional bosonic surfaces.

The new aspect that the composite structure of A_3 brings to the rheonomic action is that the surface \mathcal{M}^{11} is now allowed to be an arbitrary one *in the enlarged superspace* $\Sigma^{(528|32+32)}(s \neq 0)$ with coordinates $\mathcal{Z}^{\mathcal{N}} := (y^\nu, y^{\nu_1 \nu_2}, y^{\nu_1 \dots \nu_5}, \theta^{\tilde{\alpha}}, \tilde{\theta}'_{\tilde{\alpha}})$. With the identification $y^\mu = x^\mu$, such a surface is defined by its set of embedding functions, namely, the (already familiar) $\tilde{\theta}^{\tilde{\alpha}}(x)$ plus $\tilde{y}^{\mu_1 \mu_2}(x)$, $\tilde{y}^{\mu_1 \dots \mu_5}(x)$, and $\tilde{\theta}'_{\tilde{\alpha}}(x)$. More generally, one may define x^μ to be local coordinates of \mathcal{M}^{11} and distinguish them from the corresponding bosonic coordinates y^μ of $\tilde{\Sigma}^{(528|32+32)}(s)$ to define \mathcal{M}^{11} parametrically as

$$\begin{aligned} \mathcal{M}^{11} \subset \tilde{\Sigma}^{(528|32+32)}(s) : \quad \mathcal{Z}^{\mathcal{N}} = \tilde{\mathcal{Z}}^{\mathcal{N}}(x) \\ \Leftrightarrow \quad \begin{pmatrix} y^\mu = \tilde{y}^\mu(x) \\ y^{\mu_1 \mu_2} = \tilde{y}^{\mu_1 \mu_2}(x) \\ y^{\mu_1 \dots \mu_5} = \tilde{y}^{\mu_1 \dots \mu_5}(x) \\ \theta^{\tilde{\alpha}} = \tilde{\theta}^{\tilde{\alpha}}(x) \\ \theta'_{\tilde{\alpha}} = \tilde{\theta}'_{\tilde{\alpha}}(x) \end{pmatrix} . \end{aligned} \quad (5.20)$$

The standard $D=11$ rheonomic action S_{11}^{rh} in Eq. (2.66) with a fundamental A_3 form may

also be considered as an action for a spacetime filling 10-brane ($p = D - 1 = 10$) in standard superspace. However, some properties of this action are quite unusual for the familiar p -branes. These include the symmetry under arbitrary changes of the surface \mathcal{M}^{11} that allows us to gauge all the fermionic coordinate functions $\tilde{\theta}^{\tilde{\alpha}}(x)$ away to obtain the standard spacetime (but first order) supergravity action, the local supersymmetry with a number of parameters equal to the number of fermionic coordinate functions, and the fact that this latter symmetry still is present when the $\tilde{\theta}^{\tilde{\alpha}}(x)$ are gauged away (see [14, 15] and [32] for further discussion).

For a composite A_3 , the rheonomic action can be written as an integral over a surface \mathcal{M}^{11} in the enlarged superspace $\tilde{\Sigma}^{(528|32+32)}(s)$, Eq. (5.20),

$$\tilde{S}_{11}^{rh} = \int_{\mathcal{M}^{11} \in \tilde{\Sigma}(s)} \mathcal{L}_{11}(\mathcal{Z}^{\mathcal{N}}) = \int_{M^{11}} \mathcal{L}_{11}(\tilde{\mathcal{Z}}^{\mathcal{N}}(x)) . \quad (5.21)$$

This looks like an action of a brane which is no longer spacetime filling, as the dimension of the bosonic body of $\tilde{\Sigma}^{(528|32+32)}(s)$ superspace is 528. The functional (5.21) is of course the rheonomic action for supergravity which, due to the extra gauge symmetries, is equivalent to a spacetime (component) first order action but with a composite A_3 field. However, the fact that $\mathcal{L}_{11}(\mathcal{Z}^{\mathcal{N}})$ can be looked at as a form on the extended superspace $\tilde{\Sigma}^{(528|32+32)}(s)$ suggests trying to search for an embedding of $D = 11$ supergravity in a theory defined on $\tilde{\Sigma}^{(528|32+32)}(s)$. In particular, it is tempting to look for possible 10-brane models in $\tilde{\Sigma}^{(528|32+32)}(s)$. For instance, one might search for a superembedding condition (see [34, 43]) of the standard $\Sigma^{(11|32)}$ superspace (the worldvolume superspace of such a hypothetical brane) into $\tilde{\Sigma}^{(528|32+32)}(s)$ that could reproduce the on-shell eleven-dimensional supergravity constraints. Such a study is, however, beyond the scope of this paper.

6 Conclusions and outlook

We have studied here the consequences of a possible composite structure of the three-form field A_3 of the standard CJS $D = 11$ supergravity. In particular, we have provided the derivation of our previous result [10] by which the A_3 three-form field may be expressed in terms of the one-form gauge fields B_1^{ab} , $B_1^{a_1 \dots a_5}$, $\eta_{1\alpha}$, E^a , ψ^α associated with a *family* of superalgebras $\tilde{\mathfrak{E}}(s)$, $s \neq 0$, corresponding to the supergroups $\tilde{\Sigma}(s) = \tilde{\Sigma}^{(528|32+32)}(s)$. Two values of the s parameter recover the two earlier D'Auria-Fré [5] decompositions of A_3 , while one value of s leads to a simple expression for A_3 that does not involve $B_1^{a_1 \dots a_5}$. The supergroups $\tilde{\Sigma}(s) \otimes SO(1, 10)$ with $s \neq 0$ may be considered as nontrivial deformations of the $\tilde{\Sigma}(0) \otimes SO(1, 10) \subset \tilde{\Sigma}(0) \otimes Sp(32)$ supergroup, which is itself the expansion [10, 13] $OSp(1|32)(2, 3, 2)$ of $OSp(1|32)$. For any $s \neq 0$, $\tilde{\Sigma}(s) \otimes SO(1, 10)$ may be looked at as a hidden gauge symmetry of the $D = 11$ CJS supergravity generalizing the $D=11$ superPoincaré group $\Sigma^{(11|32)} \otimes SO(1, 10)$.

We have stressed the equivalence between the problem of searching for a composite structure of the A_3 field and, hence, for a hidden gauge symmetry of $D = 11$ supergravity, and that of trivializing a four-cocycle of the standard $D = 11$ supersymmetry algebra $\mathfrak{E}^{(11|32)}$ on the enlarged superalgebras $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$, $s \neq 0$. The generators of $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$ are in one-to-one correspondence with the one-form fields E^a , ψ^α , B_1^{ab} , $B_1^{a_1 \dots a_5}$, $\eta_{1\alpha}$. For zero curvatures these fields can be identified with the $\tilde{\Sigma}^{(528|32+32)}(s)$ -invariant Maurer-Cartan forms of $\tilde{\mathfrak{E}}^{(528|32+32)}(s)$ which, before pulling them back to a bosonic eleven-dimensional surface, are expressed through the coordinates $(x^\mu, \theta^{\tilde{\alpha}}, y^{\mu\nu}, y^{\mu_1 \dots \mu_5}, \theta'_{\tilde{\alpha}})$ of the $\tilde{\Sigma}^{(528|32+32)}(s)$ superspace.

To study the possible dynamical consequences of the composite structure of A_3 we have followed D’Auria and Fré original proposal [5] of substituting the composite A_3 for the fundamental A_3 in the first order CJS supergravity action [5, 16] (see Sec. 2.4 for a review). We have shown that such an action possesses the right number of ‘extra’ gauge symmetries to make the number of degrees of freedom the same as in the standard CJS supergravity. These are clearly symmetries under the transformations of the new one-form fields that leave the composite A_3 field invariant; their presence is related to the fact that the new gauge fields enter the supergravity action only inside the A_3 field.

We would like to mention here some similarities between the problem of searching for the composite structure of the A_3 field and the treatment of the Born-Infeld fields of D-branes and the M5-brane antisymmetric tensor field as composite fields in [9]. Born-Infeld fields are usually defined as ‘fundamental’ gauge fields *i.e.*, they are given, respectively, by one-forms $A_1(\xi)$ and a two-form $A_2(\xi)$ directly defined on the worldvolume \mathcal{W} . It was shown in [9] (see also [45]) that both $A_1(\xi)$ and $A_2(\xi)$ can be expressed through pull-backs to \mathcal{W} of forms defined on superspaces $\tilde{\Sigma}$ suitably enlarged by additional bosonic and fermionic coordinates, in accordance with the worldvolume fields/extended superspace variables correspondence principle for super- p -branes [9] (see also [23]). The embedding of \mathcal{W} into $\tilde{\Sigma}$ specifies the dynamics of the composite $A_1(\xi)$ and $A_2(\xi)$ fields. The extra degrees of freedom that are introduced by considering $A_1(\xi)$ and $A_2(\xi)$ to be the pull-backs to \mathcal{W} of forms given on $\tilde{\Sigma}$, and that produce the composite structure of the Born-Infeld fields to be used in the superbrane actions, are removed by the appearance of extra gauge symmetries [9], as is here the case for the composite A_3 field of $D=11$ supergravity. Of course, these two problems are not identical: for instance, in the case of $D=11$ supergravity with a composite A_3 , the suitably enlarged *flat* superspaces $\tilde{\Sigma}(s) = \tilde{\Sigma}^{(528|32+32)}(s)$ solves the associated problem of trivializing the CE cocycle, but a dynamical A_3 field requires ‘softening’ the $\tilde{\mathfrak{C}}^{(528|32+32)}(s \neq 0)$ MC equations by introducing nonvanishing curvatures; in contrast, the Born-Infeld worldvolume fields $A_1(\xi)$ and $A_2(\xi)$ are already dynamical in the flat superspace situation considered in [9]. Nevertheless, in both these seemingly different situations the fields/extended superspace variables correspondence leads us to the convenience of enlarging standard superspace¹³. In this way, all the fields appearing in the theory (be them on spacetime or on the worldvolume) correspond to the coordinates of a suitably enlarged superspace.

The above mentioned ‘extra’ gauge symmetries are also present in the rheonomic action for $D = 11$ supergravity when A_3 is a composite superform. The rheonomic action for a fundamental A_3 (shortly reviewed in Sec. 2.5) is derived from the spacetime component first order action just by replacing all the differential forms on spacetime by superforms on the standard superspace, pulled back to an eleven-dimensional bosonic surface \mathcal{M}^{11} . Such a surface is specified by a fermionic coordinate function $\tilde{\theta}^{\tilde{\alpha}}(x)$ which is also considered as a dynamical variable. The rheonomic action allows one, with the help of additional step of ‘lifting’ (see Sec. 2.5), to reproduce the standard superspace constraints of the $D = 11$ supergravity (see Sec. 2.2 for a discussion and 2.3 for their relation with free differential algebras). In this perspective the composite structure of A_3 allow us to consider \mathcal{M}^{11} as a surface in an enlarged superspace $\Sigma^{(528|32+32)}(s)$. As discussed in Sec. 5.3, this might indicate

¹³We further note that extended superspaces also appear in the description [40, 9] of the strictly invariant Wess–Zumino terms of the scalar branes. In a similar spirit, these invariant WZ terms trivialize their characterizing CE $(p+2)$ -cocycles [44] on the standard supersymmetry algebras $\mathfrak{E}^{(D|n)}$, including, of course, that of the $D = 11$ supermembrane, since its WZ term is given by the pull-back to \mathcal{W} of the three-form potential of the dA_3 superspace four-cocycle.

a possibility of embedding $D = 11$ supergravity in a more general theory defined in an enlarged superspace.

To summarize, the underlying gauge symmetry $\tilde{\Sigma}(s) \rtimes SO(1, 10)$ of the $D=11$ supergravity is hidden in the CJS supergravity with a fundamental A_3 , but becomes manifest in the action with a composite A_3 field. However, the latter possesses also a set of *extra* gauge symmetries, due to the fact that the new fields enter the action inside the composite A_3 field only. These extra gauge symmetries produce that the *new* gauge fields, *i.e.* the gauge fields B_1^{ab} , $B_1^{a_1 \dots a_5}$, $\eta_{1\alpha}$ corresponding to the coset $\tilde{\Sigma}(s)/\Sigma$, carry the same number of degrees of freedom as the fundamental A_3 field. In other words, the degrees of freedom in these fields that go beyond those in the fundamental A_3 are pure gauge ones. One may conjecture that these extra degrees of freedom might be important in M-theory and that, correspondingly, the extra gauge symmetries that remove them would be broken by including in the supergravity action some exotic ‘matter’ terms that couple to the ‘new’ additional one-form gauge fields. In constructing such an ‘M-theoretical matter’ action, the preservation of the $\tilde{\Sigma}(s) \rtimes SO(1, 10)$ gauge symmetry would provide a guiding principle.

A preliminary study of the local supersymmetry of the new fields shows that the preservation of the standard supersymmetry transformation rules for A_3 implies that the pure group theoretical transformation rules for the $\tilde{\Sigma}(s)/\Sigma$ gauge fields have to be modified. To analyze the structure of such a modification one can study the solution of the Bianchi identities of the $\tilde{\Sigma}(s) \rtimes SO(1, 10)$ gauge FDA, which is tantamount to studying the Bianchi identities for the superforms on $\Sigma^{(528|32+32)}(s)$ superspace.

This brings again the question of whether $D = 11$ supergravity can be included in a superfield theory defined on the enlarged superspace $\Sigma^{(528|32+32)}(s)$, $s \neq 0$, in particular in a hypothetical superfield supergravity in $\Sigma^{(528|32+32)}(s)$. A generalized supergravity in $\tilde{\Sigma}^{(528|32)} \subset \tilde{\Sigma}^{(528|32+32)}(s)$ superspace with holonomy group $GL(32)$ or $SL(32)$ was recently studied [46] (in general for $\Sigma^{(n(n+1)/2|n)}$, although with emphasis in the $n = 4, 8, 16$ cases in relation with higher spin theories in $D = 4, 6, 10$). A study of supergravity in the present $\tilde{\Sigma}^{(528|32+32)}(s \neq 0)$ superspace might lead to a different result due to the presence of the additional fermionic variables and to the natural reduction of the $GL(32)$ structure group of $\tilde{\Sigma}^{(528|32)}$ down to the $SO(1, 10)$ automorphism symmetry of $\tilde{\Sigma}^{(528|32+32)}(s \neq 0)$.

We conclude by mentioning that a possible composite structure for the A_3 field has also been considered recently [47, 48] (see also [49]) in a different perspective, in connection with the problem of anomalies in M-theory [50] and with M-theory in a topologically nontrivial situation [36, 48]. There, the A_3 field is constructed/defined using an auxiliary twelve-dimensional E_8 gauge theory. It has been asked in [48] whether the E_8 formalism is unique. In this respect it would be interesting to see whether the composite structure of A_3 field found in [5], extended to $\tilde{\mathcal{E}}(s)$ in [10], and developed in the present paper, could be useful in the context of [50, 48].

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